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ABSTRACT

Findings from five related studies carried out in Tasmania, Australia in 1979-80 are summarized. The first study attempted to determine the memory capacity of a cross-sectional population of children aged 4-7, while the second study was designed to portray differences on a variety of mathematically related developmental tasks for the same population. Data from these two studies were used to form six groups differing in cognitive capacity, and the remaining three studies each used a sample of students from these groups. The third study examined the performance and strategies used to solve a structured set of verbal addition and subtraction problems. The fourth study involved repeated assessment of performance on a set of items measuring objectives related to addition and subtraction. In the last study, the children and their teachers were observed to see how addition and subtraction were taught and whether instruction varied because of the children's cognitive capacity. Differences in capacity were reflected in performance, but instruction did not vary. The capacity of children for processing information, the procedures students have invented to solve a variety of problems, and the way in which instruction is carried out in schools did not appear to be related to each other. (Author/MNS)

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Program Report No. 83-14

Learning To Add and Subtract: The Sandy Bay Studies

by Thomas A. Romberg
and Kevin F. Collis

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November 1983

Wisconsin Center for Education Research
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SE 043 763

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LEARNING TO ADD AND SUBTRACT:
THE SANDY BAY STUDIES

by

Thomas A. Romberg and Kevin F. Collis

Report from the Project on
Studies in Mathematics

Wisconsin Center for Education Research
University of Wisconsin-Madison
Madison, Wisconsin

November 1983

Wisconsin Center for Education Research

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Abstract

This monograph summarizes the findings from five related studies carried out by the authors in Sandy Bay, Tasmania, Australia, in 1979-80. The overall purpose of the studies was to examine whether children who differed in cognitive capacity learned to add and subtract in different ways.

The first study was designed to determine the memory capacity of a cross-sectional population of children. The second study was designed to portray differences on a variety of mathematically related developmental tasks for the same population of children. Data from these two studies were used to form groups of children who differed in cognitive capacity. Six groups were formed via cluster analysis, with memory capacity being the primary distinguishing characteristic.

The third, fourth, and fifth studies each used a sample of students from the six cluster groups across grades. The third study examined both the performance and the strategies these children used to solve a structured set of verbal addition and subtraction problems. The fourth study involved repeated assessment of the children's performance on a set of items measuring objectives related to addition and subtraction. Finally, in the last study these children and their teachers were observed during classroom instruction in mathematics to see how addition and subtraction were taught and whether instruction varied because of the children's cognitive capacity.

Children's differences in capacity are reflected in their performance on both verbal and standard problems and in the strategies they

use to solve problems. However, instruction does not vary for these children within classrooms. The picture which emerges is one of children struggling to learn a variety of important concepts and skills. Some children are limited by their capacity to handle information. Most are able to solve a variety of problems by using invented strategies, those which have not been taught. They dismiss or fail to see the value of the taught procedures in solving those problems. Finally, the capacity of children for processing information, the procedures students have invented to solve a variety of problems, and the way in which instruction is carried out in schools are not related to each other.

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Chapter 1

AN INTRODUCTION TO THE SANDY BAY STUDIES

> For several centuries, being able to find "one's sums and differences" has been considered one mark of a schooled person. Although today we may have expanded our expectations about what constitutes literacy, we still expect all children to efficiently carry out operations on whole numbers. Yet, in spite of these expectations about the skills of addition and subtraction, there has been little consensus about how such skills develop. (Romberg, 1982, p. 1)

The basic question under investigation was Do children who differ in cognitive processing capacity learn to add and subtract differently? In raising this question, we assumed that the evolution of children's performance on addition and subtraction tasks must be related both to their developing cognitive abilities and to their engagement in related instructional activities.

The five studies reported in this monograph represent an attempt to draw together data gathered from four different perspectives on this question. Each perspective is viable in its own right. However, our intent was to see whether in combination they could better portray how addition and subtraction skills develop. The first approach, from the classical individual differences perspective, was to use psychometric techniques in two studies to identify students with differing cognitive capacities. The second approach, from the cognitive processing perspective, was to gather clinical interview data over time about the strategies children use to solve verbal addition and subtraction problems. The third approach, from the quasi-experimental perspective, was to use an achievement monitoring procedure to assess pupil growth. The final approach, the direct instruction teaching perspective, was to use a time-on-task observation procedure to determine the features of classroom instruction. Each approach taken separately tends to examine the phenomena under consideration from a microscopic point of view. Our intention was to incorporate the data from the different sources into one picture, since all were derived from the same sample of children, with a view to seeing the phenomena macroscopically. By looking at the same phenomena from different perspectives at the same time, insights might be gained which would not be revealed by a single lens approach, no matter how powerful the lens.

The rationale for integrating data from these four perspectives was set out in a conceptual paper (Romberg, Carpenter, & Moser, 1978). In this paper the authors described how, for nearly a decade (1968-1976) the Studies in Mathematics project at the Wisconsin Center for Education Research had concentrated its efforts on the relationship between instructional processes, methods, and materials and the acquisition of mathematical skills associated with mathematical learning. The work in this project led to the development of a complete elementary mathematics program, Developing Mathematical Processes (DMP) (Romberg, Harvey, Moser, & Montgomery, 1974, 1975, 1976). Although DMP was based on the available empirical evidence and theories of learning, development, and instruction (see Romberg, 1977), a number of questions were raised as the program was being developed.

Many assumptions had to be made. These assumptions were related both to the way children learn mathematics and to the way in which teachers really acted in the classroom. With regard to the former, it was assumed that instruction could be matched to how learners developed mathematical concepts. A constructivist position on learning had been taken based on notions from a Piagetian cognitive-development model (Lovell, 1972). In particular, Lovell had argued that

> alongside the abstraction of the mathematical idea from the physical situation, there must be an introduction of the relevant symbolism and the working of examples, involving drill and practice and problems on paper. (Lovell, 1972, p. 177)

This led to assumptions that pupils should work in small groups on activities that involved physical materials, students should have opportunity to talk, and so forth (Lovell, 1972). Although these assumptions seemed to be logical extensions of developmental theories, there was little hard evidence about how specific number skills actually develop in children. Furthermore, the implications of these notions to instruction lacked empirical support. Teachers clearly needed more than a brief outline and a few suggestions to make a workable instructional plan. Moreover, the ways in which teachers grouped children and interacted with them clearly played a significant part in what the children did. This last implication, in the final analysis, would seem to determine the outcome of instruction.

In short, it became clear that a more complete picture was required to describe children's cognitive capacity with mathematical material; to characterize the mathematical material to be learned; and to identify the features of classroom instruction such as how children perform on instructional tasks, teacher actions in relation to the presentation of mathematical material, pupil engagement, and teacher-pupil interactions during lessons. Thus, from past work, it was clear that the interactions between capacity, content, and instruction needed to be carefully examined.

Cognitive Capacity

Many researchers in mathematics education have indicated the necessity to take heed of the child's intellectual abilities when designing curriculum units and instructional techniques. The problem is that "although students bring life to mathematics, they add to the instructional complexity for they also bring to the activities the full range of their differences" (Romberg, 1977, p. 85). One must have criteria about which differences are to be considered. The criteria used in this study were based on claims from two sources. The first was from the extensive work of a number of educational psychologists in the Thurstone tradition of distinct mental abilities. From test scores and psychometric analyses, these psychologists identified differential abilities, traits, aptitudes, styles, and so forth. For example, such characteristics as intelligence, rate of learning, field independence/dependence, and spatial ability have been identified, and samples of students ordered from high to low on those traits. These are assumed to be fixed, stable characteristics, biological in origin, which describe intellectual differences between individuals in the same way as height, weight, stature, and so forth describe physical characteristics. Although we did not utilize tests developed from this perspective, we used the psychometric strategy of administering to each student a number of tests, scoring the tests, and then relating the scores.

From work in cognitive development, we chose the measures to be used in the study. This perspective is based on information that individuals adaptively interact with the environment and gradually evolve intellectual processes through discontinuous stages. Rather than being fixed, differences between individuals are viewed as a function of growth. Children in the primary grades, for example, usually are at a "concrete-operations" stage, think in terms of themselves (are egocentric), and think of concrete referents near at hand. Hence, they should not be expected to reason about hypothetical, external situations.

The choice of tests from this perspective grew out of the work on children's understanding of mathematics. This research gained impetus following the failure of the "new math" programs to live up to their early expectations. Psychologists interested in mathematics learning began to investigate developmental and learning phenomena by using elementary mathematical material as a vehicle (e.g., Collis, 1975). These investigators used the clinical interview as a technique for studying the concepts which children had formed in relation to their experience with mathematics. Much of the work was stimulated by the provocative notions of Jean Piaget (Inhelder & Piaget, 1958). Later interest was related to the work on memory capacity by Pascual-Leone (1976) and Case (1972). This view of cognitive functioning enabled psychologists to turn from the mere description of stages of development of mathematical thinking to an explanation of the phenomena which kept appearing in their work with individual children.

This evolution can be traced through the work of Collis (1971, 1974a, 1974b, 1975, 1976, 1978, 1980a, 1980b, 1982; Collis & Biggs, 1979; Biggs & Collis, 1982). The earlier papers use mathematical items to describe and, to some extent, to modify Piaget's stage theory

(Inhelder & Piaget, 1958). The later papers, after about 1976, provide tentative explanations of the developmental phenomena found earlier in terms of Case's information processing theory (Case, 1975). The most recent papers (e.g., Biggs & Collis, 1982) describe a response model which, while allowing for the stage phenomenon, places the emphasis on the increasing complexity of responses within a given stage.

In summary, a psychometric strategy was adopted to determine cognitive processing capacity. The initial task was to find and administer measures of cognitive functioning which appeared logically related to the learning of mathematical material and which seemed to be related to the children's level of cognitive development. We selected instruments which could be shown prima facie to contain tasks related to early mathematical learning such as number conservation and counting. We gave two batteries of tests. The first battery of tests was designed to measure the memory capacity (M-space) of the child for processing mathematical material. The second battery included tests constructed to measure the child's level of cognitive development on dimensions from the Piagetian model, such as conservation and transitivity, and presumably related to mathematical ability.

We then used psychometric procedures, factor analysis, and cluster analyses to interpret the data from both batteries and to group children. From this approach, we assumed that well defined sets of children with specific cognitive characteristics could be identified.

Content to be Learned

We decided to examine children's early work in addition and subtraction for several reasons. First, this area represents the first attempt that the school makes to teach what might be recognized as formal mathematics. Second, a lot of work had been done at the Wisconsin Center (e.g., Moser, 1979) on logically analyzing the semantic-syntactic relationship for these mathematical skills as they apply at the early elementary school level. Third, various researchers had identified several strategies used by young children to solve elementary addition and subtraction problems when they were presented in either physical or verbal mode (see Carpenter, Moser, & Romberg, 1932). Finally, a clinical observation schedule for assessing performance on some addition and subtraction tasks had recently been developed (Carpenter & Moser, 1979).

To solve a typical addition and subtraction problem, one first must understand its implied semantic meaning. Quantifying the elements of the problem comes next (e.g., choosing a unit and counting how many). Then, the implied semantics of the problem must be expressed in the syntax of addition and subtraction. Next the child must be able to carry out the procedural (algorithmic) steps of adding and subtracting. Finally, the results of these operations must be expressed. Children bring to such problems well developed counting procedures, some knowledge of numbers, and some understanding of physical operations, such as "joining" and "separating," on sets of objects. Thus, from this context researchers have a unique opportunity to examine variations in how children process information prior to, during, and after formal instruction. Studying how children learn to solve such problems is the phenomenon addressed in this book.

Five basic addition and subtraction semantic forms were identified by Moser (1979):

1. joining--the process of putting together two representations which have an attribute in common to form a single representation with the same attribute;
2. separating--the reverse of "joining" in that it represents the taking apart of one representation with a particular attribute to make two representations with the same attribute as the original;
3. difference--the process of determining how much more (or less) the larger (or smaller) representation is than the other;
4. part-part-whole--a static relationship that exists between a representation having a particular attribute and two of its component parts that are disjoint from each other; and
5. equalizing--the process of making two representations the same on a particular attribute.

To build the connection between a semantic form and relevant symbolism, one form was used as a model to introduce the symbolism. Given that there are many semantic forms for which the same symbolic sentence is appropriate, the pedagogical problem is how to relate the symbolism with the semantic problems. Traditionally, the symbolism has been taught independent of word problems. Then often the symbolic procedures were taught, and some word problems were assigned so that students could apply their symbolic procedures. No serious consideration was given to the semantic structure of the problems. It is no surprise, then, that for different types of problems students found little connection between the problems and the symbolic procedures they had been taught (e.g., Vergnaud, 1982).

When developing DMP, we recognized this problem and decided to use one semantic context to introduce and give meaning to the symbolism and then to relate the symbolism to other semantic situations. In the initial version of DMP, equalizing was used. This context proved to be difficult for both teachers and students when examining other semantic forms. A revised set of materials was later developed in which part-part-whole was used as the basic context (Kouba & Moser, 1979, 1980).

The strategies identified by scholars (see Carpenter, Moser, & Romberg, 1982) included the following:

1. use of knowledge of "basic number facts" whether simple ("How many children if there are 4 boys and 5 girls?") or complex, when the calculation required a rather sophisticated application of a known number fact together with some mental manipulation of the numbers involved;
2. use of representation skills which usually involved the modeling of a number given in a verbal mode with objects or pictures;
3. use of counting--counting all, counting on, or counting back--all with or without use of physical or pictorial representation involving one-to-one correspondence.

Of significant interest to researchers and teachers must be the link, if any, between the logical analysis of the semantic forms of problems and the strategies children actually use to solve such problems.

Several points were noted at the outset of this project. First, many children seemed to have developed the "primitive" or "child" strategies prior to school learning experiences or at least prior to formal instruction on consolidated "efficient" methods of solution. Second, the logical analysis of the operations would seem to imply that these initial strategies would become more and more inefficient as the number of semantic forms is increased, or the numbers become larger, or the number of steps necessary for solution increases. Thus, it would seem to be a reasonable goal of mathematics instruction to teach more formal, generalizable algorithmic procedures for solving the variety of addition and subtraction problems.

However, little is known about several aspects of this process and a number of questions arise. How will learning of the mathematical procedures be affected by the degree, type, and success of the "primitive" problem-solving strategies possessed by the individual child? How do the successful children finally put together their "primitive" strategies with the formal mathematical modes of presentation? How does the teacher adapt instruction to take account of the child's demonstrated level of functioning in this area? Raising these questions leads to a consideration of the relationship of general cognitive functioning to performance by children on such problems.

Classroom Instruction

Although the focus of this investigation was on the cognitive capacity of children and how they solve verbal problems, the children are being taught to add and subtract in school. To capture some aspects of classroom instruction, we decided on a strategy that involved data gathered on a sample of students at each of three grade levels (grades 1, 2, and 3). It is at these grades that addition and subtraction skills are taught. The sample of students was to reflect differences in cognitive capacity.

Second, data on the performance of the students would be collected using an achievement monitoring battery recently developed (Buchanan & Romberg, 1983). This battery provides information on a variety of aspects of adding and subtracting, and in several administrations profiles of growth can be obtained. The profiles then can be used as indicators of the effectiveness of instruction.

Third, we decided to observe teacher actions, pupil actions, and teacher-pupil interactions for those children at each grade level who differed in cognitive capacity. The proposition that "teachers make a difference" has been central to much of the previous work done on mathematics education at the Wisconsin Center for Education Research. For example, the steps in the IGE Instructional Programming Model (Klausmeier, 1977) are all descriptions of actions teachers are to take. In addition, as DMP was being developed, a set of behaviors was specified for teachers to use in teaching the program. Despite these efforts, little evidence is available to substantiate the importance of teacher actions. Berliner (1975) probably pointed to the reasons for this lack when he identified a long list of problems facing researchers who attempt to examine the relationships between teacher behaviors and pupil performance. He saw methodology as one major category of impediments to progress in the area.

The primary methodological problems that Berliner identified were the inadequate framework for the conceptualization of teacher tasks and the assumed direct relationship between teacher tasks and pupil performance. It is possible that the logical analyses of the problem, subsequent to Berliner's rather pessimistic overview, are as far from the realities involved in the classroom situation as the analyses carried out on mathematics curriculum programs in the 1950s and 1960s or the logical application of general psychologists' theories of development and learning to mathematics programs of a decade ago. Perhaps what is needed is a fresh look at the problem.

In this study we decided to concentrate attention on the teachers' actions as they related to children of known cognitive characteristics and, moreover, on the same children's reactions related to the teachers' initiating action. The approach should make some move toward both assisting in conceptualizing the teachers' real tasks and testing the notion that the teacher has some discernible effect on the pupil's performance.

Another major criticism made by Berliner was the lack, at that time, of instruments that gave the researcher a clear understanding of the meaning of data gathered by objective tests or surveys. Moreover, even when observational techniques were employed, it was not usual to code pupil actions. We decided to take advantage of recent advances in research tools in this area by using the instrument developed for the study of instructional time with DMP (Romberg, Small, Carnahan, & Cookson, 1979). This instrument takes into account the behavior of both teachers and children.

Conclusion

The studies were designed not only to gather and analyze data on the four perspectives set out above, but also to examine the interactions between the data sets. Obviously, there are a number of interactions which would be of considerable interest, but in view of our interests and to break some new ground we decided to concentrate on the interaction between the child's cognitive processing capability and the other data sets.

We first identified a sample of children aged 4-8 years with specific cognitive characteristics. Sample selection required measuring M-space (Study 1) and measuring cognitive development (Study 2) of a population of 4- to 8-year-olds. Next, we studied the mathematics performance, strategies used, and instruction provided the sample over a three-month period. In clinical interviews the children's performance and strategies were determined with a set of verbal addition and subtraction problems (Study 3). Achievement was measured with a set of standard written addition and subtraction tasks (Study 4). The nature of the instruction provided and children's actions and engagement were determined in classroom observations (Study 5).

We assumed that from this set of studies we would be able to relate performance at a given time (in terms of level achieved and strategy adopted) to the child's cognitive capability and to a specific set of instructional activities the child has been engaged with. In this way, we could consider various questions about change in performance and strategy and their possible causes.

The various research techniques used, the data gathered, and their analysis are described in the next four chapters. Chapter 2 is concerned with the means we used to characterize the cognitive processing capabilities in Studies 1 and 2. Chapters 3 and 4 relate the cognitive level of each group to addition and subtraction problems.

In chapter 3 the individual clinical interview data coded for both performance and strategies used by the child is presented (Study 3). In chapter 4 achievement on paper-and-pencil tests of addition and subtraction is presented. In chapter 5 we attempt to relate cognitive level to teacher-pupil interactions. Chapter 6 provides a summary of the findings and some conclusions which draw together the understandings obtained through the studies and suggests some direction for further research.

In 1979 the Research Committee of the Graduate School at the University of Wisconsin-Madison, the Wisconsin Center for Education Research, and the University of Tasmania jointly funded the principal investigators to carry out the proposed series of studies relating children's cognitive capacity to their performance and to the strategies they used when working addition and subtraction problems.

Chapter 2

IDENTIFICATION OF GROUPS OF CHILDREN WHO DIFFER IN COGNITIVE PROCESSING CAPABILITIES

In this chapter the classification of children into groups according to their cognitive processing capabilities with mathematical materials is presented. Cognitive processing capability is a derived categorization label based on a combination of measures of working memory capacity (M-space) and measures of the level of cognitive development as determined by the Piagetian model. The M-space measures were the basis of the classification of children into categories, while the developmental tests give an indication of developmental criteria which are applicable within each category.

The Population

All of the children in Sandy Bay Infant School in Hobart, Tasmania, were tested for this study. The school is located on the Derwent River in Sandy Bay, a suburb of Hobart near the University of Tasmania. The community is middle to upper-middle class. Table 1 gives details about the age/grade characteristics of the sample and number of children involved.

Test Administration

A research assistant and two experienced teachers were hired to administer the tests. All were trained before the testing proceeded. One interviewer administered the Counting Span test; a second the Mr. Cucui test; and the third the Digit Placement and the Backward

Table 1

Age/Grade Characteristics of Population

Characteristic		Class and Grade						
		1 K-AM	2 K-PM	3 Prep	4 Gr. 1	5 Gr. 1/2	6 Gr. 2	Total
Number	Boys	16	11	8	8	15	15	73
	Girls	9	9	13	14	9	12	66
	Total	25	20	21	22	24	27	139
Age	Youngest	4.9*	5.0	5.4	6.2	6.5	7.3	
	Oldest	5.1	5.7	6.1	7.3	7.10	8.2	
	Average	4.11	5.4	5.10	6.7	7.3	7.8	

*4.9 means 4 years 9 months as of October 1, 1979.

Digit Span tests. Children were randomly selected by their teacher to come to the interview room and randomly assigned to an interviewer. Most children took two tests on one day and the other two a day or two later. All testing was completed within ten days.

Study 1--M-space

Information processing theories are based on the idea that mental functions can be characterized in terms of the way information is stored, accessed, and operated upon. Mental structures are discussed in terms of an intake register through which information from the environment enters the system, a working or short-term memory (M-space) in which the actual information processing occurs, and a long-term memory in which knowledge is stored.

The working memory's growing capacity to process information appears as a fundamental characteristic of cognitive development in a number of theories (Bruner, 1966; Case, 1978a; Flavell, 1971). Young children are quite limited in their ability to deal with all the information demands of complex tasks. Their limited capacity seems to be a critical developmental factor which constrains learning in instructional situations (Case, 1975, 1978a, 1978b).

Pascual-Leone (1970, 1976) proposed a theory which operationalizes the development of this information processing capacity or M-space. According to this theory, learning is a change in behavior resulting from factors extrinsic to the psychological system. Learning produces a change in the repertoire of schemes (internally represented behavioral units or patterns) available to the subject. Since M-space is limited, the number of information chunks which can be coordinated to produce a new scheme is limited. Therefore, the complexity of schemes learned is also limited; the processes of learning are constrained by the developing psychological system. Pascual-Leone's theory is concerned with the functional aspects of development and the mental processing of information. Learning through instruction depends on the child's capacity to process all of the essential incoming information.

To generate hypotheses about children's performance on specific tasks, both the information processing capacity (M-space) of the child and the information processing demands of the task must be known. This study addresses the problem of assessing information processing capacity.

The rationale for giving a set of different tests to measure the construct is based on the results of two recent studies, one by Hiebert (1979), in which the measure of M-space (Backward Digit Span) proved not to be predictive of learning mathematical skills, and another by Case and Kurland (1978) in which three different measures of M-space (Counting Span, Mr. Cucui, and Digit Placement) were given. Although in Case and Kurland's study, positive correlations (.50 to .60) were found between the three tests, the consistency between the measures was not high. Recent work by Case and associates (Case, Kurland, Daneman, & Emmanuel, 1979) suggests that it may be very difficult to construct any one general measure of M-space which will predict performance on a wide range of tasks. Their data indicate that task variables may be more important than previously supposed in determining the M-space demands of a particular task. Thus, we decided to use the three tests from Case and Kurland's study along with the Backward Digit Span test from Hiebert's study to see if together they would yield an estimate of a child's M-space. The tests chosen also seemed appropriate in terms of the task variables involved in learning to add and subtract.

Procedures

The Tests

Counting Span. This test was developed by Case and Kurland (1978). Conceptually, it is straightforward. The operation required is counting. The items which must be stored are the products of a series of counting operations. Children are presented with a sequence of arrays of geometric shapes to count and are asked to recall the number of objects in the arrays preceding the current trial as soon as they have finished counting the shapes on the current stimulus card. The number of arrays in the set is incremented from trial to trial and the child's M-space is assumed to be equal to the maximum number of arrays which he or she can count while maintaining perfect recall.

The test includes 33 items. However, at most, only five items were scored at any one of five M-space levels. To reduce the total number of trials a modified "ceiling basal" method was used (Bachelder & Denny, 1977). Children were presented with sets from different M-space levels until it was determined at what level they passed and at what level they failed. They were then presented with a larger number of trials until the level of complete success and the level of complete failure had been determined.

Mr. Cucui. This measure was designed in Pascual-Leone's laboratory by DeAvila, for use with children with an imperfect command of

English (DeAvila & Havassy, 1974). It was quick to administer and suitable for use with four-year-olds as well as older children.

On each trial, children are presented with the outline of Mr. Cucui. After viewing it for five seconds, they are told to remember what parts of his body are colored. They are then presented with a blank outline drawing of Mr. Cucui and told to point to the parts which were colored. There are 25 items, five different items at each of five levels; a level is defined as the number of body parts which are colored.

This test is the only one which does not require the student to count or use numbers. Instead, recall of spatial location is required to respond correctly. The ceiling-basal method was followed for the Counting Span test.

Digit Placement. This is a measure of M-space which was developed and standardized by Case. It is known to yield the same norms as other tests of M-space (cf. Case, 1972), and to load highly on the general factor defined by more lengthy M-tests such as Pascual-Leone's CSVI (Case & Globerson, 1974). The basic procedure is to present subjects with a set of numbers. The first n - 1 of these are in ascending order of magnitude and the nth is out of order (e.g., 2, 5, 9, 12, 7). After the numbers have disappeared from view, the children are asked to indicate where the final number belongs in the original series. M-space corresponds to the maximum set size for which the task can be executed successfully. For this test, there are 15 items, five for each of three levels; levels 1 and 5 as measured in the two tests above are not tested. All items were given to each subject.

Backward Digit Span. The form used in this study was developed by Hiebert (1979). On each trial, the experimenter reads a series of digits. The subject is to repeat them in the reverse order. M-space corresponds to the maximum series size correctly repeated. In this test, there are 40 items (10 at each of four levels; level 1 as measured in the first two tests is not tested and all items were given to each student.

Scoring the Tests

Although each item could obviously be scored correct or incorrect and the total correct counted to estimate each child's M-space level, there were at least two sound reasons why this procedure would be inadequate. First, since sets of items in each test were designed to measure different levels of M-space item scores would need to be weighted to reflect those levels--especially as two of the tests did not aim to measure all five levels. Second, since the "ceiling basal" procedure was used with two of the tests, some items were not actually administered to each child; items not administered but at a level lower than where the child responded correctly were scored correct and all items at a level higher than where the child responded correctly were scored incorrect.

Four scoring rules were devised for each test. The full details regarding those are available in Romberg and Collis (1980a) and will not be reported here. The scoring method S_3 was finally deemed the most satisfactory, and was used in analysis, and is used for the discussions in this chapter. The general rule for scoring the tests was as follows:

$$S_3 = 1 + Z_1 + Z_2 + . . . z_n$$

where n = the number of levels represented in the test

and

where $Z_i = 1$ if 4/5 or 5/5 items were correct, otherwise

$Z_i = 0$.

Do not score Z_{i+1} if $Z_i \neq 1$.

The only difference was for the Backward Digit Span test for which the following line was substituted:

where $Z_i = 1$ if 8/10, 9/10, or 10/10 items were correct.

Results

Table 2 shows the frequency of scores (M-space level) for children in each class and for the total population for each test. In addition, class means and standard deviations are presented.

The basic distributions of scores for the four memory tests provide two interesting results. First, although older children generally have higher scores, the overlap of scores among children at different grade levels is quite striking. Scores are clearly age-related but do not appear to be specifically determined by age. Second, the variation of M-space level for individual children across tests (note variation in within-class frequencies across tests) could imply that the context of the text may give students a cue which helps them answer questions. In addition, if partial level scores are allowed for children answering items on a test at a higher level than they can be credited with under scoring procedure S_3, it is a reasonable deduction, on the evidence from the protocols, that the move from one level of M-space to another is gradual.

Table 2

Frequency of Scores for the M-Space Tests

Class	Score							
	0	1	2	3	4	5	X̄	SD
Counting Span Test								
(1) K-AM		22	3				1.12	.33
(2) K-PM	1	9	9	1			1.50	.69
(3) Prep		11	9	1			1.52	.60
(4) Gr. 1		4	15	3			1.96	.57
(5) Gr. 1/2		1	15	8			2.29	.55
(6) Gr. 2		2	12	12	1		2.44	.70
Totals	1	49	63	25	1	0	1.83	.75
Mr. Cucui Test								
(1) K-AM		12	12	1			1.56	.58
(2) K-PM		5	11	2			1.75	.64
(3) Prep		4	12	5			2.05	.67
(4) Gr. 1		1	9	8	4		2.68	.84
(5) Gr. 1/2			6	9	7	2	3.21	.93
(6) Gr. 2		1	6	7	11	2	3.26	1.02
Totals	0	25	56	32	22	4	2.45	1.05
Digit Placement Test								
(1) K-AM		24		1			1.08	.40
(2) K-PM		20					1.00	.00
(3) Prep		19	2				1.10	.30
(4) Gr. 1		18	3	1			1.23	.53
(5) Gr. 1/2		12	6		6		2.00	1.25
(6) Gr. 2		5	1	2	19		3.30	1.20
Totals	0	98	12	4	25	0	1.68	1.17
Backward Digit Span Test								
(1) K-AM		13	12				1.48	.51
(2) K-PM		2	18				1.90	.31
(3) Prep		1	20				1.95	.22
(4) Gr. 1			18	4			2.18	.40
(5) Gr. 1/2			16	8			2.33	.48
(6) Gr. 2			8	15	4		2.85	.66
Totals	0	16	92	27	4	0	2.14	.64

Relationship of Scores on the Tests

Each of the tests, it was hoped, would reflect the amount of M-space available to the children with early math related material. However, the tasks are different; the student population covered a wide age/grade range; and children's scores demonstrated considerable variation in performance. Thus, it was important to investigate with some care whether the different tests yielded similar classifications of children. Three statistical procedures were performed on the data: (1) a correlation matrix was set up to show the correlations between the scores from the four tests for the total population; (2) the data for all pairs of tests were cross tabulated to see how many classifications were the same; and (3) a factor analysis was performed on the correlation matrix to determine the dimensionality of the scores.

Correlations of test scores. The correlations (see Table 3) while all positive and statistically significant, are not particularly high. The highest is only .64. It seems clear that different tests will not necessarily classify children into the same M-space levels.

Cross tabulations of scores for the four tests. To examine the similarity between classification schemes based on the four tests, we cross tabulated the data for each test with each other test. The proportion of students who were classified into the same categories and into different categories in each comparison is shown in Table 4. The percentage of individuals who were differently classified in the comparisons range from 68% to 46%.

This cross tabulation demonstrates that the tests classify children in different ways. If these different classifications are along a single dimension, there is not a serious problem; it would

Table 3

Correlations of Scores for the Four Memory Tests

Test	CS	MC	DP	BDS
Counting Span (CS)	1.00			
Mr. Cucui (MC)	.49	1.00		
Digit Placement (DP)	.61	.50	1.00	
Backward Digit Span (BDS)	.52	.40	.64	1.00

Table 4

Number and Percentage of Classifications Which are the Same, Higher for the First Test, and Lower for the First Test for all Test Comparisons

	Test Comparisons (A/B)					
Classi-fication	CS/DP N(%)	CS/MC N(%)	CS/BDS N(%)	DP/MC N(%)	DP/BDS N(%)	MC/BDS N(%)
Same (A=B)	58(42)	47(34)	75(54)	49(35)	44(32)	57(41)
Higher (A>B)	36(26)	16(12)	13(9)	19(14)	31(22)	55(40)
Lower (A<B)	45(32)	76(55)	51(37)	71(51)	64(46)	27(19)

Note: CS = Counting Span
DP = Digit Placement
MC = Mr. Cucui
BDS= Backward Digit Span

mean that each test identifies different cutoff points on this one dimension. However, if these tests are found to measure more than one dimension, then different things are being measured by each test.

Factor analysis. The results of the cross tabulation made the question of examining the dimensionality question more critical. A factor analysis was performed on the correlation matrix presented in Table 3 for the four tests across the total population. The model used was a multifactor solution model. All extractions were principle factor extractions with iterative estimates of commonalities, and the varimax rotation procedure was used. The data from this factor analysis appear in Table 5. A single factor was extracted. However, it should be noted that Mr. Cucui test did not load heavily on this factor, and a considerable amount of the variance is still unaccounted for. The Mr. Cucui test is the only one of the four which does not ask children to count, suggesting that the factor is a quantitative M-space factor involving memory of number or counting sequences. The Mr. Cucui test, on the other hand, requires memory of spatial orientation.

In summary, the four tests measure one primary factor, quantitative M-space. Thus, to classify children into M-space levels, it would seem best to administer a combination of tests as was done in this study and then classify the children with regard to that underlying structure. No one test alone, it appears, could reliably classify individuals into an M-space level. The next section indicates that a classification made on the basis of the results of perhaps three tests should be fairly reliable for most individuals.

Table 5

Factor Analysis for the Four Memory Tests

	Factor
	1
Eigenvalue	2.59
% variance	64.8
Raw (rotated) factor matrix	
Counting Span	.44(.56)
Digit Placement	.54(.72)
Mr. Cucui	.30(.37)
Backward Digit Span	.44(.51)

Cluster Analysis

Since the factor analysis showed that one dimension accounted for nearly 2/3 of the variance, it seemed desirable for the next stage of the project to classify the children in the population along this single dimension. A cluster analysis procedure, which uses Euclidian distances between points,[1] was used for the classification.

This analysis indicated that there were six viable groups. Table 6 gives the estimated group vectors for six groups identified. In the analysis, the last four groups (3, 4, 5, and 6) were closer together than Groups 1 and 2. This suggests that Groups 1 and 2 are distinct and that Groups 3, 4, 5, and 6, while different from each other, are less distinct.

Group 1 is largest with 59 members. For the tests separately, the levels for this group are CS, level 1; DPT, level 1; BDS, level 1; and MC, level 1. This group is clearly at M-space level 1, the lowest M-space level in the domain being measured. Only for Backward Digit Span could some children be placed at level 2.

Group 2 has 38 members. The levels for this group are CS, level 2; DPT, level 1; BDS, level 2; and MC, level 2. These children exhibit a basic M-space level 2. They are below that level on the Digit Placement test and nearly reach level 3 on the Mr. Cuci test. These differences seemed, from the protocols, to be due to contextual

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[1] The usual Euclidian distance between points in four dimensions was used, i.e.,

$$d = \sqrt{(x_1 - y_1)^2 + (s_2 - y_2)^2 + x_3 - y_3)^2 + (x_4 - y_4)^2}$$

Table 6

Estimated Vector for the Six Groups Derived from a Cluster Analysis Where the Distance Between Score Vectors is Less than 1.50

Group	Amalgamated distance	Number of children	Test				Overall M-space classification
			CS	DPT	BDS	MC	
1	1.05	59	1.32	1.07	1.73	1.61	1
2	1.44	38	1.90	1.66	2.13	2.76	2
3	1.43	16	2.25	2.10	2.25	3.69	2S+
4	1.03	11	2.91	4.00	2.91	2.46	3S-
5	1.06	4	2.17	3.83	2.67	4.50	3S+
6	1.23	6	2.50	4.00	3.75	3.75	4S-

factors: Ss found the instructions for DPT more complex than the others; and either spatial perception or ability to check information contributed to scores on Mr. Cucui.

Group 3, with 16 members, scored slightly above level 2 on three tests and nearly reached level 4 on the Mr. Cucui Test. Either their spatial perception is quite high or they are able to chunk information on that test, but they still exhibit a basic M-space level of 2. We have labeled this group level 2S+ to highlight the fact that these children are above that level on the one spatial test.

Group 4 has 11 members. On two tests, CS and DBS, children are at level 3; on DPT they are at level 4, but on the Mr. Cucui test, they are only at level 2. Their basic M-space level is probably level 3. The spatial perception involved in Mr. Cucui appears not be be as highly developed at their quantitative abilities. Therefore, we have classified them 3S-.

Group 5 has only 4 members, who have a similar pattern of levels to those in Group 4 except they score very well on Mr. Cucui. Their basic pattern seems to place them at M-space level 3, and therefore we have classified them 3S+.

Group 6 has 6 members. They are basically at M-space level 4. They are at that level on three tests but score below level 3 on Counting Span. It is not clear what the discrepancy on this test implies, and the protocols did not assist in this case. It could, of course, be simply a sampling/testing variation especially since the numbers in the category are so small. This variation needs closer examination than we were able to perform in this study. However, this group is lower than Group 5 on the Mr. Cucui test, but overall their quantitative skills are at level 4. Therefore, we have labeled them 4S-.

Overall, these results suggest an underlying cognitive mechanism. The contextual setting, number or spatial orientation (quantitative or qualitative) has a significant effect on the child's ability to respond on any given occasion. This shows possible significant problem solving strategy/instruction reception differences between groups with the same basic cognitive processing potential. One could hypothesize that spatial development (qualitative) and number development (quantitative) strategies appear to be interwoven and occur close together in time, but some children achieve number skill prior to spatial skill and others vice versa.

Study 2--Cognitive Development

The reason for wanting a battery of tests which measure cognitive development is rooted in the theory of Jean Piaget (1974). For Piaget, cognitive development is embedded in a developing human system. The development of cognition is inseparable from the growth of biological and psychological faculties. Development is a broad-based process, generalizing to a wide variety of situations.

Piaget's position is summarized in the following statement:

> I think that development explains learning, and this option is contrary to the widely held opinion that development is a sum of discrete learning experiences. (1974, p. 176)

The phrase "development explains learning" implies that the outcome of a learning experience is in part accounted for by developmental capabilities. That is, learning potential is defined (or explained) to a large extent by developmental level.

For this project a battery of 10 tests was devised, all impinging on the early development of the child's ability to work with elementary quantitative and logical concepts concerned with pre-mathematical skills. We tested the whole population to relate developmental characteristics to characteristics already derived from the M-space tests.

The Cognitive Development Tests

As stated earlier, the choice of specific tests was based on our intent to examine the relationship of cognitive capability to children's performance on addition and subtraction tasks.

Of the 10 tests, 7 were selected from a large battery of tests constructed by Fullerton (1968), 2 from tests devised by Romberg, Carpenter, and Moser (1978), and one was constructed by the authors for this study. Details of each of the tests can be found in Romberg and Collis (1980b).

Extension (E). This group test was developed by Fullerton (1968). Children are to decide which of three choice boxes has the same number of dots as a sample box. The term extension refers to the fact that the number sets extended beyond the usual level of subitemization to a higher portion of the number scale. The test contained 12 items. The number of correct responses was scored. A correct answer was interpreted to mean that the child was able to set up a one-to-one correspondence between sets.

Ordinal Correspondence (OC). In this group test, also developed by Fullerton (1968), the format for the items was similar to those in the Extension test. This test also contained 12 items. The number of correct responses was scored. A correct answer was interpreted as meaning that the child was able to establish an ordinal correspondence between sets.

Conservation of Number (Wohlwill, CN-W). This group test, also developed by Fullerton (1968), was based on an earlier test developed by Wohlwill (1960). Six items were given. The number of correct responses was scored. A correct response was interpreted to mean that the child was able to preserve one-to-one correspondence between sets after one set had been rearranged (i.e., was able to overcome perceptual distractions).

Addition-Subtraction (Wohlwill, AS-W). The items for this group test, also developed by Fullerton (1968) and based on Wohlwill's earlier work (1960) were interspersed with those of the previous test (CN-W) because of the similarity between the two tests. This test differed only in that a single object was either added to or subtracted from the collection of objects in front of the children. In this case a correct response was interpreted to mean that the child was able to recognize that an increase or decrease in one of two sets in one-to-one correspondence means these sets are no longer in such correspondence. Six items were given and the number correct scored.

Transitivity (T). The authors developed this six-item group test because the Coordination of Relations Equivalence test (CRE, described next) requires a child to attend to both transitivity and a linear rearrangement of sets. The present test was designed to assess just transitivity. A correct response was interpreted as the child being able to preserve both equivalence and order relationships. A total correct score was recorded for each child.

Coordination of Relations of Equivalence test (CRE). This six-item group test was developed by Fullerton (1968). The items are similar to those in the transitivity test except that the fixed set is also transformed (lengthened, shortened, or heaped together). A correct response here was interpreted as the child being able to preserve equivalence relationships even after rearrangement. The same scoring procedure was used as for transitivity (T).

Class Inclusion (CI). This individually administered test of two items was developed by Romberg, Carpenter, and Moser (1978). A correct response was interpreted as a child being able to logically subdivide a set into distinct subsets.

Additive Composition of Number (ACN). This individually administered test, developed by Fullerton (1968), includes three items which ask children to respond to three quite different composition tasks. A correct response implies the child can establish an equivalence relationship by the common practice of sharing and preserve such a correspondence when distracting information is presented.

Counting On (CO). This individually administered test was developed by Romberg, Carpenter, and Moser (1978). The test includes three items for each of the three levels of counting on; small number onto a number less than 10, small number onto a number between 10 and 20, and a large number onto a number between 10 and 20. The typical question asked was "Could you start counting at 13 to find the number that is 4 more than 13?" Children were marked as passing a level if two of three items were answered correctly. A total score was then recorded of the number of levels passed (0, 1, 2, or 3).

Counting Back (CB). This test is like formal CO; however, in this case the typical question asked was, "Could you count back starting at 15 to find the number that is 4 less than 15?" The scoring procedure used was the same as in CO.

Test Administration

Because the order in which these tests were administered was important, and because they would be administered to children of varying ages, two decisions were made to gather the data more efficiently. First, the tests were separated into four sets to be administered at separate times. Second, all of the tests were given to all children. The organization of the tests and the rules for selecting who was to take which test are given in Table 7. The interview tests and set 2 were given to all children. A child passing the two tests in set 2 (CN-W and AS-W) was assumed to have passed set 1 and was given set 3. However, if a child failed either of the tests in set 2, set 1 was administered, and the child was assumed to have failed set 3.

On the interview tests one assistant administered the Counting On (CO) and Counting Back (CB) tests, and the other administered the Class Inclusion (CI) and the Additive Composition of Number (ACN) tests. Again children were randomly selected by their teacher to come to the interview room (the teachers' lounge). Each interviewer was in a corner of the room. Children were randomly assigned to an interviewer. Children took two tests on one day and the other two a day or two after. Shortly after the interviews were completed the group batteries were given. Set 2 was given first to groups of 6-8 children at a time from each class. The research assistant presented the

Table 7

Tests Included in Each Set, Sequence of Administration, and Rules for Selecting Subjects

Order	Set (tests)	Rule
1	Interview (ACN, CI, CO, CB)	All children
2	Set 2 (CN-W, AS-W)	All children
3	Set 1 (E, OC)	Children failing either test in set 2
4	Set 3 (T, CRE)	Children passing both tests in set 2

stimulus information for each test following a script and using a large magnet board. The other assistants observed the children to make sure they were working on the correct page, responding in the right place, and not copying from others. Set 1 was given next, followed by set 3. All testing was completed within four weeks.

Results

Full summary tables for the raw score data for each of the tests is given by Romberg and Collis (1980b). To examine the relation between the tests and the structure of the battery itself, a two-step procedure was followed.

Fullerton (1968) used scalogram analysis to organize the battery of tests he developed. He found tests which grouped together, and he established an order for the tests based on test difficulty. Unfortunately, that methodology fails to establish the underlying dimensionality of the data matrix or the possible structure of the assumed hierarchy. A more satisfactory method is to determine first the dimensionality of the intercorrelations of the tests. If the matrix is unidimensional, then a hierarchy can be established.

The intercorrelations across the whole population for the 10 cognitive processing tests appear in Table 8. The correlations are all positive but fairly low, ranging from .24 to .79; 17 of the 28 correlations fall between .40 and .58. We decided to exclude the E and OC tests from the correlation matrix for further analysis on the grounds that they were baseline tests on which most children scored at the ceiling.

Table 8

Intercorrelations of the Ten Cognitive Development Tests

	E	OC	ACN	CN-W	AS-W	CI	CO	CB	T	CRE
E	1.00									
OC	.45	1.00								
ACN	.22	.25	1.00							
CN-W	.30	.32	.35	1.00						
AS-W	.35	.37	.48	.51	1.00					
CI	.13	.13	.32	.24	.28	1.00				
CO	.22	.28	.55	.43	.42	.44	1.00			
CB	.15	.21	.49	.40	.39	.45	.79	1.00		
T	.13	.16	.43	.42	.36	.47	.52	.61	1.00	
CRE	.17	.21	.51	.55	.48	.39	.58	.62	.68	1.00
Maximum	12	12	3	6	6	2	3	3	6	6
Mean	10.88	10.63	1.97	4.86	5.03	.51	1.35	1.06	3.93	1.75
Std. deviation	1.90	2.28	.94	1.56	1.34	.81	1.29	1.21	4.68	1.49

To determine the dimensionality of the intercorrelations, a factor analysis was performed on the matrix shown in Table 8 with tests E and OC excluded. A multifactor solution model was used. All extractions were principal factor extractions with iteration estimates of commonalities; the varimax rotation procedure was employed. The results of this analysis are shown in Table 9.

A two factor solution was derived, although the Eigenvalue for the first factor is considerably larger than that for the second factor. An examination of this rotated factor matrix shows that the counting tests (CB, CO) load heaviest on the rotated first factor followed by the tests in Battery 4, T and CRE. This factor may reflect a mature level of counting skill. The four other tests also load on this factor but not to the same degree. At best we can say that it is probably a quantitative factor influenced by the ability to count. The second factor seems more qualitative, involving the ability to make comparisons and see transformations without having to count. In particular, the Wohlwill tests (AS-W and CN-W) load heaviest on this rotated factor load. One test, Class Inclusion, does not load heavily on either factor. Since Class Inclusion involves logical reasoning and is the only nonquantitative test, this finding gives credence to the interpretations given to the first two factors.

The factor analysis of the data seemed to show that there were two interpretable dimensions underlying performance on the tests. However, since the first factor accounted for such a large proportion of the variance (54.30%), we examined the possible hierarchical ordering of the tests using Guttman's (1954) simplex procedures. It

Table 9

Factor Analysis for Eight Cognitive Process Tests

	Factors	
	1	2
Eigenvalue	4.34	.92
% variance	54.30	11.50
Raw (rotated) factor matrix		
ACN	.64(.46)	.07(.45)
CN-W	.61(.25)	.35(.65)
AS-W	.60(.24)	.37(.66)
CI	.52(.49)	-.13(.23)
CO	.80(.76)	-.22(.33)
CB	.83(.85)	-.33(.26)
T	.73(.60)	-.06(.41)
CRE	.81(.56)	.11(.59)

is clear as one examines the correlation matrix as a whole that the tests are not in simplex order. Even when we take a subset of the matrix, the five tests (ACN, CN-W, CI, T, and CRE) which might be considered to test aspects of logical functioning at this level, the criteria are not satisfied. It seems from all the evidence then that there is no basis for a hierarchical ordering of these tests. In summary, the cognitive development tests do not seem to measure a single dimension. Rather they measure two discernible dimensions, a quantitative counting factor and a qualitative correspondence factor.

The purpose of this study was to discern the cognitive development levels of the children in the population in relation to a battery of tests which tested developmental variables presumed to underlie the early development of mathematical skills. The results show that about two-thirds of the variance on the tests can be explained in terms of two dimensions, a quantitative factor influenced by the ability to count, which accounts for over half of the variance, and a qualitative factor which involves an ability to make comparisons and see transformations without counting.

The Cognitive Processing Capability Categories

In this section of the analysis, we attempted to combine the information from the M-space tests and the cognitive development tests with a view to grouping the individuals into categories which have distinct describable cognitive characteristics.

To begin with, a correlation matrix (Table 10) was drawn up for the four M-space tests and the eight cognitive development tests (tests E and OC being omitted for reasons given earlier). The

Table 10

Correlations of the Eight Cognitive Processing Tests and the Four M-Space Tests

M-space tests	Cognitive Processing Tests							
	ACN	CN-W	AS-W	CI	CO	CB	T	CRE
CS[a]	.54	.32	.39	.43	.63	.61	.47	.53
DP[b]	.54	.44	.41	.45	.77	.79	.69	.63
MC[c]	.46	.32	.37	.48	.53	.55	.46	.47
BDS[d]	.48	.48	.50	.38	.61	.58	.55	.54

[a] Counting Span

[b] Digit Placement

[c] Mr. Cucui

[d] Backward Digit Span

correlations range from .29 to .79, with 20 of the 32 falling between .40 and .59. The higher correlations with the M-space tests occur with both the counting tests (CO, CB). This is not surprising. The counting tests undoubtedly require a larger memory capacity than some of the other tests. However, there is no apparent significant variation in correlations of the different memory tests with the cognitive processing tests. This suggests that the positive correlation is along a single dimension.

To check this suggested unidimensional relationship, a factor analysis was carried out in which the four M-space tests were added to the eight cognitive development tests. The data for that factor analysis appear in Table 11. Again, as was the case with the factor analysis of the cognitive development tests (see Table 9), two factors appeared. The two factors have the same structure as the two factors that appeared in the earlier analysis. The memory tests load on the first factor but not the second.

At this point, we decided that we had enough information to look for a pattern in the achievement on all cognitive tests for each of the six groups formed by the cluster analysis of the M-space tests. The proportion correct in each cognitive test for each M-space category is set out in Table 12; a graphical representation of the same information is shown in Figure 1.

It can be seem that there are clear differences between Groups 1 and 2 and the other four groups. Within the latter groups, Groups 3 and 4 differ little from each other but from Groups 5 and 6 who are also very similar.

Table 11

Factor Analysis for Eight Cognitive Development Tests
and the Four M-Space Tests

	Factors	
	1	2
Eigenvalue	6.52	1.02
% variance	54.40	8.50
Raw (rotated) factor analysis		
ACN	.65(.56)	.08(.36)
CN-W	.58(.40)	.41(.50)
AS-W	.59(.37)	.41(.60)
CI	.55(.56)	-.12(.12)
CO	.83(.78)	-.18(.28)
CB	.84(.85)	-.25(.15)
T	.73(.73)	-.01(.16)
CRE	.78(.70)	.16(.31)
CS	.71(.68)	-.13(.25)
DP	.86(.74)	-.19(.10)
MC	.63(.68)	-.09(.25)
BDS	.73(.62)	.13(.43)

Table 12

Percent Correct for the Six M-Space Groups on the Ten Cognitive Development Tests

	Test									
			Factor 1 (Quantitative)							
Group (M-space level)	Baseline		Factor 2 (Qualitative)							Logical
	E	OC	AS-W	CN-W	ACN	CO	CB	CRE	T	CI
1(1)	90	84	46	48	45	9	3	6	2	6
2(2)	100	100	91	71	80	52	35	43	14	20
3(2S+)	100	100	87	93	71	82	78	73	53	50
4(3S-)	100	100	100	90	87	93	87	80	90	50
5(3S+)	100	100	100	100	100	91	75	100	100	88
6(4S-)	100	100	100	100	93	100	87	100	80	90

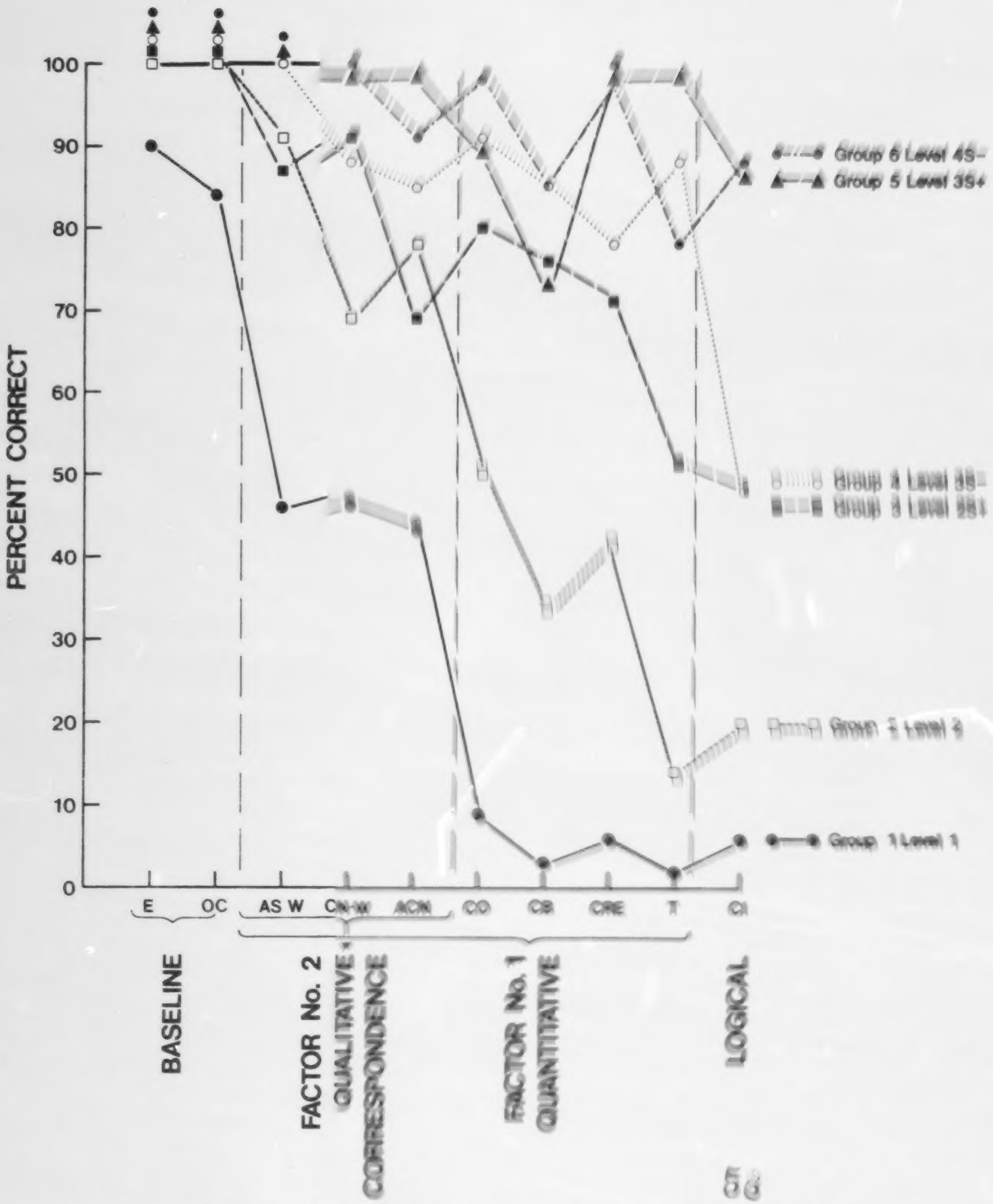

Figure 1. Pattern of scores (percent correct) for the six M-space groups on ten cognitive process tests grouped by factors.

Group 1 children with M-space level 1 are below the other groups in all four areas and are in general incapable of handling quantitative tasks. They are capable of handling qualitative comparisons and transformations only at a moderate level.

Group 2 children with M-space level 2 are also without specific quantitative skills, although they performed considerably better than Group 1 on all the tests. They can handle qualitative correspondence at an acceptable level although they scored somewhat lower than the other groups on the conservation of number test.

Group 3 children with M-space level 2S+ are high on qualitative correspondence; have developed the specific counting skills of counting on and counting back; but are inadequate in their use of these skills on the transitive reasoning test. They also are inadequate on logical reasoning although considerably better than Groups 1 or 2 on that test.

Group 4 children with M-space level 3S- are high on qualitative correspondence and all the quantitative tests, but inadequate on the logical reasoning test. In fact they differ significantly from Group 3 only on the additive composition test and the transitivity test.

Groups 5 and 6 with M-space levels 3S+ and 4S- present similar profiles on these tests. They reach the ceiling on the qualitative correspondence tests, scoring a little higher than Groups 2, 3, and 4. Like Group 4 children, they have very high scores on all the quantitative tests. Children in these groups are high on the class inclusion test.

From these cluster groups a sample of students was drawn for studies 3, 4, and 5 in this series in the following school year.

Conclusion

Based on data from four memory tests and eight cognitive development tests, we have been able to identify groups of children who have well defined by different cognitive processing capabilities. This was accomplished in the following steps. First, using cluster analysis on the memory test scores, we identified six groups of students with similar patterns of responses. Second, from the results of a factor analysis, we found the tests loaded on two factors: a quantitative factor that involves mature counting strategies and a qualitative correspondence factor. Third, by examining how the six groups defined by the M-space analysis performed on the cognitive tests we demonstrated that the cognitive processing scores of five of those six groups differed systematically from each other.

This last step was the basis for the remainder of the project. We formed five distinct groups of students (cluster groups 5 and 6 were combined) with known cognitive capabilities related to the learning of mathematical materials. In the following chapters, we describe how this information was used to study several aspects of the children's interaction with mathematics in early elementary school.

In conclusion the data gathered and analyzed in this chapter suggest that the following propositions deserve close attention by both researchers and practitioners:

1. A global qualitative/quantitative distinction is apparent in children's mathematical thinking in the early school year;

2. M-space level seems to be related to the development of other cognitive skills;
3. The suggested developmental sequence in the preschool to early elementary years in mathematically related reasoning appears to be: comparison -- qualitative correspondence -- quantitative -- logical operations;
4. An M-space level of 1 is enough for handling simple comparison tasks;
5. An M-space level of 2 is enough for qualitative correspondence and is a prerequisite for the development of number skills;
6. An M-space level of 3 seems necessary for success on sophisticated counting tasks.

In all, these data suggest simple correspondence (both equivalence and order) appears to be the first ability to develop. This is followed by a qualitative correspondence capacity which involves understanding how correspondence between two sets is preserved or changed under varying circumstances. Next, the quantitative skills of counting on and counting back develop, followed by their use in transitivity tasks. Finally, the capacity for logical reasoning develops.

Chapter 3

COGNITIVE PROCESSING CAPACITY AND CHILDREN'S PERFORMANCE ON VERBAL ADDITION AND SUBTRACTION PROBLEMS

In this chapter, the third study in this set is reported. Its purpose was to study the relationships of children's capacity to their performance and their use of strategies on verbal addition and subtraction problems. The importance of knowing how children learn the concepts of procedures of addition and subtraction should be self-evident. Also, it is frequently assumed that children must first master computational skills and then begin to solve verbal addition and subtraction problems. However, it has been clearly demonstrated that children develop a variety of strategies for solving such mathematical problems independent of instruction (c.f., Carpenter & Moser, 1979; Ginsburg, 1977; Resnick, 1978). In fact, many of the strategies they use are more sophisticated and demonstrate more insight than the procedures that are taught.

A sample of the children tested in the previous studies (Chapter 2) and selected to reflect different cognitive capabilities were clinically interviewed on three occasions over a three-month period in 1980 (February 27-29, April 9-11, and May 26-28). In each interview, a set of verbal addition and subtraction problems was given to each student. Each child's performance and strategies were coded by the interviewer.

The Sample for this Study

The children from the earlier studies had been advanced a grade in school since previously tested. Furthermore, the grade 2 students who were in Sandy Bay Infant School in October now were in grade 3 and in different primary schools. Most, however, were enrolled at Waimea Heights Primary School.

Our intent was to have a sample of two to four students from each cognitive level in each grade. We began with rosters of students from each grade and their cognitive level. Then an initial selection of students was made. However, after school began, some third graders originally in one class were switched to another. This created some imbalance across classes but should not have effected the results. The students by cognitive group and class in this study are shown in Table 13.

Interview Tasks

An interview consisted of six problem types (tasks) given under four of six conditions. The six types included two problems solvable by addition of the two given numbers and four problems solvable by subtraction of the two given numbers. The types differ in terms of their semantic structure. The semantic characterization for these six problem types is detailed in Moser (1979) and in Carpenter and Moser (1979).

Table 14 presents representative problems in the order in which the problems were administered to the children. The actual wording for each problem type differed but the semantic structure remained constant. Within each problem, two of three numbers from a number triple (x, y, z) defined by $x + y = z$, $x < y < z$, were given. In the

Table 13

Children in Each Cluster Group in Each Class

	Sandy Bay Infant School		Waimea Heights Primary School			
	Class		Class			
	1	2	3	4	5	
Cognitive Group	Grade 1	Grade 2	Grade 3	Grade 3	Grade 3	Total
1	3	2	0	0	0	5
2	3	6	0	4	0	13
3	1	2	2	3	3	11
4	0	0	2	3	3	8
5,6	0	0	3	1	3	7
Totals	7	10	7	11	9	44

Table 14

Problem Types

Task	Sample Problem
1. Change/Join (Addition)	Pam had 3 shells. Her brother gave her 6 more shells. How many shells did Pam have altogether?
2. Change/Separate (Subtraction)	Jenny had 7 erasers. She gave 5 erasers to Ben. How many erasers did Jenny have left?
3. Combine/Part Unknown (Subtraction)	There are 5 fish in a bowl. 3 are striped and the rest are spotted. How many spotted fish are in the bowl?
4. Combine/Whole Unknown (Addition)	Matt has 2 baseball cards. He also has 4 football cards. How many cards does Matt have altogether?
5. Compare (Subtraction)	Angie has 4 lady bugs. Her brother Todd has 7 lady bugs. How many more lady bugs does Todd have than Angie?
6. Change/Join, Change set Unknown (Subtraction)	Gene has 5 marshmallows. How many more marshmallows does he have to put with them so he has 8 marshmallows altogther?

two addition problems x and y were presented, with the smaller number x always given first. In the four subtraction problems, z and the larger addend y were presented. The order of presentation of y and z varied among problem types.

The six semantic problem types used were presented under six conditions, although not all children responded to all conditions. Four conditions result from crossing smaller zs and larger zs with presence and absence of manipulative materials. In the smaller number problems (called B problems), the addition guideline of $5 \leq z \leq 9$ was imposed. In the larger number problems (called C problems), the restriction on the sum was $11 \leq z \leq 15$. Problem sets Bp and Cp were given with manipulatives present; the same sets given with manipulatives absent were called Ba and Ca.

For the interviews with third-grade children, the domain of 2-digit numbers was included. In the 2-digit domain, two subdomains were identified. In the D problems, no regrouping (borrowing or carrying) is required to determine a difference or sum when a computational algorithm is used. In the second subdomain, E problems, regrouping is required. For the 2-digit problems, the sum z is restricted to numbers in the 20s and 30s. All third-grade children took the C, D, and E problems. Complete details of the procedures used are reported in Romberg, Collis, and Buchanan (1981).

Interview Method

Three trained interviewers administered the interviews (see Cookson & Moser, 1980, for details of interviewer training procedures and reliability). One interviewer worked at Sandy Bay Infant School and the other two at Waimea Heights Primary School. Each interviewer was able to conduct 8 to 12 interviews a day, depending on the schools' schedules and on the task level. (The C tasks took longer than the B tasks.) At the schools, the interviewers were assigned interview areas, which were quiet rooms separate from distracting activities. The verbal tasks were read and reread to the child as often as necessary so that remembering the given numbers or relationships caused no difficulty. An individual interview required two sessions, one for the B tasks and the other for C tasks (or one for the C and the other for D and E). The sessions lasted 15-25 minutes each, with each child receiving the same sequence of problems. No child was interviewed twice in one day.

Coding Subject Responses

All of the possible codings of student responses are presented in detail in Cookson and Moser (1980). Three or four elements were coded for each child: model used, correctness, strategy, and, if incorrect, error. A record of each subject's response to the tasks were compiled from the coding sheets. These profiles are the basis for all other statistical information appearing in this chapter and are reported in Romberg, Collis, and Buchanan (1981).

Data Aggregation and Analysis

The interview data have been summarized in terms of percent correct and general strategy. The data for percent of items answered correctly by children are summarized by examining the differences for children with differing cognitive processing capabilities. It was anticipated that children in groups 5 and 6 would answer more items correctly than those in group 4, who in turn would answer more items correctly than the group 3 children, and so forth.

Pupil strategy was categorized according to type of model used (if any), strategy or process used, and errors (if any). Five general categories for the B and C problems are the following:

1. Direct modeling. Use of the manipulatives provided, or fingers, to stand for the problem entities. Actions performed on the objects generally correspond to the action or relationship described in the problem.
2. Use of counting sequences. Use of the string of counting words, either forward or backward, where the entry point in the sequence is a number other than "one." Counting may proceed in either direction a given number of counts, or until a desired number (usually one of the numbers given in the problem) has been reached. This requires a second counting of some sort of a tracking mechanism, often aided by the use of fingers.
3. Routine mental operations. Use of memorized number facts by direct recall.
4. Nonroutine mental operations. Derivation of a nonmemorized fact through manipulation of some other recalled fact. As an example, the fact for 6 + 8 can be derived by determining it to be two more than the easily remembered "doubles" fact of 6 + 6.
5. Inappropriate Behaviors. Guessing, using one of the given numbers in the problem, adding instead of subtracting, or giving no answer at all.

For the D and E data, the five categories used with the B and C tasks were used if students did not write a sentence. If students did write a sentence, three other categories were used.

6. Correct sentence/algorithmic. This category of behavior includes the standard algorithms taught in school as well as any "invented" (Carpenter & Moser, 1982) ones that involve considerations of place value. Algorithmic behavior must be exhibited by use of paper and pencil.
7. Correct sentence/non-algorithmic. After writing a sentence, the work is done mentally as was frequently seen in problems in which no regrouping (D tasks) was required (Moser, 1981).
8. Inappropriate sentence. This behavior involves writing and working the wrong sentence (e.g., addition instead of subtraction).

Details of what specific model, strategy, and error data were used to form these categories are presented in Romberg, Collis, and Buchanan (1981).

The plan for analyses of the aggregated data was based on the two primary dimensions in this study--differences in the level of problem administered and differences in children's cognitive capacity. The problem dimension involves a completely crossed repeated assessment (three interviews) of six problem sets (Bp, Ba, Cp, Ca, D, and E) with six tasks in each set (combine/join, combine/separate, and so on). The student dimension involves children nested in cognitive levels within classes and in turn, within grades.

The data matrix is incomplete since grade 1 and grade 2 children did not take the D and E problems, the grade 3 children did not take the B problems, and not all cognitive levels are represented in each grade level. The small number of subjects, the unequal cell sizes, and the extensive incompleteness of the matrix have limited us to describing the frequencies and testing a few of the differences with chi-square statistics.[1]

For purposes of this report, frequency and percent correct and frequency of use of strategy are presented for children with different cognitive processing capabilities. The data are presented for three problem sets (B, C, and D, E combined) and for each semantic task within each set. Other analyses were performed for each interview and by grade level but are not reported here. Those analyses can be found in Romberg, Collis, and Buchanan (1981).

Problem Sets by Cognitive Groups

To examine whether differences in cognitive capacity are reflected in different percentages of correct responses, separate tables are presented for each problem set. In Table 15, the data for the B problems which were given only to grade 1 and grade 2 children clearly show that there is a significant increase in percent correct

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[1] Because of the large number of trials and the lack of a systematic plan to test differences, an alpha level of .01 was arbitrarily chosen to test significance. In addition, tests which yielded probability values between an alpha of .05 and .01 ($.05 > p > .01$) were considered marginally significant. All χ^2 values were calculated via 2 x 2 contingency tables where frequency of correct answers or strategy was dichotomized.

Table 15

Frequency and Percent Correct by Cognitive Group for All Level B Tasks

			Correct Responses	
Cognitive Group	N	Total Responses[a]	Frequency	Percent
1	5	180	100	56
2	9	312[a]	235	75
3	3	108	95	88
4	-	-	-	-
5,6	-	-	-	-
Total	17	600	430	72

[a] When all children were present for all 3 interviews, number of trials equals N times 12 problems (6 Bp and 6 Ba) times 3 occasions.

(56% to 75% to 88%) for children in cognitive groups 1, 2, and 3, respectively ($\chi^2 = 47.19$, $p < .01$).

For the C problems given to all children, the percent correct for children in different cognitive groups is shown in Table 16. The differences are striking. The group 1 children only got 22% correct while children in groups 5 and 6 got 96% correct. There is significant increase from group 1 to group 2 (22% to 65%, $\chi^2 = 94.38$, $p < .01$), from group 2 to group 3 (65% to 81%, $\chi^2 = 26.74$, $p < .01$), and again from group 4 to groups 5 and 6 (83% to 96%, $p < .01$). The lack of difference in percent correct between cognitive group 3 and group 4 children is not surprising since these groups differed very little on the cognitive tests.

For the D and E problems given only to grade 3 children, the pattern of correct responses were very similar. Thus, for summary purposes, the data on these problems have been combined in Table 17. For these students, the difference between percentage correct for children in cognitive groups 2 and 3 (49% and 67%) is significant ($\chi^2 = 11.76$, $p < .01$) as are the differences between cognitive group 4 and cognitive group 5, 6 children (62% and 83%, $\chi^2 = 30.05$, $p < .01$). Again, the differences in performance between cognitive groups 3 and 4 on both sets of problems are not significant.

Overall, our predictions about percentage of the items answered correctly were found to be accurate, except that children in cognitive capacity groups 3 and 4 differed very little in terms of their overall performance.

Table 16

Frequency and Percent Correct by Cognitive Group for All Level C Tasks

Cognitive Group	N	Total Responses[a]	Correct Response	
			Frequency	Percent
1	5	180	40	22
2	13	456	206	65
3	11	396	320	81
4	8	264	220	83
5,6	7	252	241	96
Total	44	1548	1117	72

[a] When all children were present for all 3 interviews, number of trials equals N times 12 problems (6Cp and 6 Ca) times 3 occasions.

Table 17

Frequency and Percent Correct by Cognitive Group for All Level D,E Tasks

			Correct Response	
Cognitive Group	N	Total Responses[a]	Frequency	Percent
1	-	-	-	-
2	4	144	71	49
3	8	264	176	67
4	8	252	155	62
5,6	7	252	210	83
Total	27	912	612	67

[a] When all children were present for all 3 interviews, number of trials equals N times 12 problems (6D and 6E) times 3 occasions.

Tasks by Cognitive Group

Within each problem set, one item representing each of six tasks (change/join; change/separate; combine/part unknown; combine/whole unknown; compare and change/join; change/set unknown) was given. Because the different semantics of each problem type elicit different cognitive demands, it was anticipated performance would vary with the tasks. Following Greeno's (1980) categorization for the six tasks given (see Table 14) we expected Tasks 1 and 2 (change/join and change/separate) to be the easiest, for they demand only a change/cause schema; Task 4 (combine/whole unknown) to be next in difficulty, for it involves a harder combination schema; Tasks 6 and 3 (the missing addends problem) should follow in difficulty because of the location of the missing information; and Task 5 (comparison/subtraction) to be hardest because it involves a comparison schema which requires more units of memory to handle. The percent correct data for each cognitive group for each task in the B set of problems are presented in Table 18. The pattern of differences between cognitive groups is consistent with group 3 children performing better than group 2 who, in turn, perform better than the group 1 children. As expected for the B level, Tasks 1 and 2 were easy for all children. Tasks 4 and 6, however, were just as easy. Task 3 was more difficult, and Task 5 was hard for all children.

The percent correct data for each cognitive group on each task for the C set of problems are presented in Table 19. If two-thirds of the items were correct, then this was used as a rough criteria for success for this data. Again, a consistent pattern of the higher cognitive group children getting as many or more items correct is

Table 18

Frequency and Percent Correct by Cognitive Group for Each B Task

			Correct Response	
Cognitive Group	N	Total Responses	Frequency	Percent
Task 1 Change/Join (+)				
1	15	30	23	77
2	26	52	44	85
3	9	18	18	100
4	–	–	–	–
5,6	–	–	–	–
Total	50	100	85	85
Task 2 Change/Separate (-)				
1	15	30	21	70
2	26	52	42	81
3	9	18	16	89
4	–	–	–	–
5,6	–	–	–	–
Total	50	100	79	79
Task 3 Combine/Part Unknown (-)				
1	15	30	12	40
2	26	52	40	77
3	9	18	16	89
4	–	–	–	–
5,6	–	–	–	–
Total	50	100	68	68

			Correct Response	
Cognitive Group	N	Total Responses	Frequency	Percent
Task 4 Combine/Whole Unknown (+)				
1	15	30	21	70
2	26	52	43	83
3	9	18	16	89
4	–	–	–	–
5,6	–	–	–	–
Total	50	100	80	80
Task 5 Compare (-)				
1	15	30	7	23
2	26	52	22	42
3	9	18	12	67
4	–	–	–	–
5,6	–	–	–	–
Total	50	100	41	41
Task 6 Change/Join, Change set unknown (-)				
1	15	30	16	53
2	26	52	44	85
3	9	18	17	94
4	–	–	–	–
5,6	–	–	–	–
Total	50	100	77	77

Table 19

Frequency and Percent Correct by Cognitive Group for Each C Task

Cognitive Group	N	Total Responses	Correct Response: Frequency	Correct Response: Percent
Task 1 Joining (+)				
1	15	30	9	30
2	38	76	55	72
3	33	66	57	86
4	22	44	40	91
5,6	21	42	40	95
Total	129	258	201	78
Task 2 Separating (-)				
1	15	30	7	23
2	38	76	52	68
3	33	66	51	77
4	22	44	34	77
5,6	21	42	40	95
Total	129	258	184	71
Task 3 PPW, missing addend (-)				
1	15	30	6	20
2	38	76	54	71
3	33	66	56	85
4	22	44	33	75
5,6	21	42	41	98
Total	129	258	190	74
Task 4 PPW (+)				
1	15	30	11	37
2	38	76	52	68
3	33	66	53	80
4	22	44	38	86
5,6	21	42	40	95
Total	129	258	194	75
Task 5 Comparison (-)				
1	15	30	1	3
2	38	76	30	39
3	33	66	49	74
4	22	44	38	86
5,6	21	42	39	93
Total	129	258	157	58
Task 6 Joining, missing addend (-)				
1	15	30	6	20
2	38	76	53	70
3	33	66	54	82
4	22	44	37	84
5,6	21	42	41	98
Total	129	258	191	74

apparent. The one exception to this pattern was on Task 3 (combine/part unknown), the group 4 children do not do as well as the group 3 children on those tasks. Group 1 children are generally unable to work any of the C problems successfully. The majority of group 2 children work all problems except Task 5. Children in higher groups are able to work all problems. However, except for the difficult comparison problems (Task 5), the tasks were of equal difficulty.

The same data for the D and E sets of problems are shown in Table 20. And again, the same pattern is evident except for the cognitive group 4 children whose performance is marginally lower than group 3 children on Tasks 1 and 2 and is about the same as group 2 children on Task 5. Overall, group 2 children are only successful on Task 1. Group 3 and 4 children are successful on Tasks 1, 4, and 6. And, group 5, 6 children are successful on all tasks. However, unexpectedly, Task 2 was as hard as Tasks 3 and 5 for the whole population. What these data suggest is that when problems have large enough numbers that children should use algorithms, the implied computational procedures become more important than the semantics. Thus, addition problems are easier than subtraction problems. While Task 6 is a subtraction problem, it is often solved using additive notions, making it easier than Task 2.

In summary, although there are important variations in performance due to problem set (size of number) to specific task, and to grade, it is clear that children who have been identified as having different cognitive processing capabilities consistently perform

Table 20

Frequency and Percent Correct by Cognitive Group for Each D,E Task

			Correct Response	
Cognitive Group	N	Total Responses	Frequency	Percent
Task 1 Change/Join (+)				
1	-	-	-	-
2	12	24	16	67
3	22	44	39	89
4	21	42	33	78
5,6	21	42	37	88
Total	76	152	125	82
Task 2 Change/Separate (-)				
1	-	-	-	-
2	12	24	10	42
3	22	44	27	61
4	21	42	21	50
5,6	21	42	31	74
Total	76	152	89	58
Task 3 Combine/Part Unknown (-)				
1	-	-	-	-
2	12	24	9	38
3	22	44	22	50
4	21	42	22	52
5,6	21	42	35	83
Total	76	152	88	58

			Correct Response	
Cognitive Group	N	Total Responses	Frequency	Percent
Task 4 Combine/Whole Unknown (+)				
1	-	-	-	-
2	12	24	14	58
3	22	44	31	70
4	21	42	31	74
5,6	21	42	39	93
Total	76	152	115	76
Task 5 Compare (-)				
1	-	-	-	-
2	12	24	12	50
3	22	44	28	64
4	21	42	20	48
5,6	21	42	30	71
Total	76	152	90	59
Task 6 Change/Join, Change set unknown (-)				
1	-	-	-	-
2	12	24	10	42
3	22	44	29	66
4	21	42	28	67
5,6	21	42	38	90
Total	76	152	105	68

differently on these addition and subtraction tasks regardless of the other important factors.

Strategies Used by Children

As outlined in the first part of this chapter, the data on strategies used by children have been summarized in terms of five categories for the B and C problem sets (direct modeling, counting sequences, routine mental operations, nonroutine mental operations, and inappropriate) and eight categories for the D and E problem sets (the same five no-sentence categories as for B and C tasks plus correct sentence-algorithmic, correct sentence-non-algorithmic, and incorrect sentence).

We expected that children with low cognitive capacity would either use inappropriate strategies or directly model problems. Children at a higher capacity level would then use counting sequences followed by routine mental operations and algorithms in increasing frequency for children at higher levels of competency.

To examine whether children with different levels of cognitive capacity use different strategies, separate tables are presented for each problem set. For the B problems given only to grade 1 and 2 children (Table 21), as expected, there was a significant increase in use of routine mental operations (8% to 27% to 35%) for children with higher cognitive capacity ($\chi^2 = 36.97$, $p < .01$) and a corresponding significant decrease in use of an inappropriate strategy (39% to 18% to 7%; $\chi^2 = 34.80$, $p < .01$). However, unexpectedly, the frequency of use of the other categories remained constant over cognitive levels.

For the C problems given to all children, the strategy data for children in differing cognitive groups are shown in Table 22. The

Table 21

Frequency of Use of Strategies by Cognitive Group and Category for All 8 Tasks

		Direct Modeling		Counting Sequences		Routine Mental Operation		Nonroutine Mental Operation		Inappropriate	
Cognitive Group	Responses	Frequency	Percent	Frequency	Percent	Frequency	Percent	Frequency	Percent	Frequency	Percent
1	180	69	38	20	11	15	8	5	3	71	39
2	312	120	38	43	14	85	27	8	2	56	18
3	108	39	36	17	16	39	36	5	5	8	7
Total	600	228	38	80	13	139	23	18	3	135	22

Table 22

Frequency of Use of Strategies by Cognitive Group and Category for All C Tasks

		Direct Modeling		Counting Sequences		Routine Mental Operation		Nonroutine Mental Operation		Inappropriate	
Cognitive Group	Responses	Frequency	Percent	Frequency	Percent	Frequency	Percent	Frequency	Percent	Frequency	Percent
1	180	50	28	1	0+	2	1	0	0	127	70
2	456	166	36	82	18	59	13	27	6	122	27
3	396	71	18	130	33	104	26	38	10	53	13
4	264	30	11	79	30	92	35	38	14	25	9
5,6	252	32	13	101	40	105	42	14	6	0	0
Total	1548	349	22	393	25	362	23	117	8	327	21

picture here is more dramatic. As anticipated, children in cognitive group 1 either directly model the problems (28% of the trials) or use an inappropriate strategy (70% of the trials). Use of an inappropriate strategy goes down consistently with cognitive group (70% for group 1 children to 0% for group 5, 6 children). Direct modeling is the strategy most often used by cognitive group 2 children; counting sequences by group 3 children; and routine mental operations by groups 4 and 5, 6 children who also used counting sequences frequently.

For the D problems, which were taken only by the third grade children, the strategy data are summarized in Table 23. As expected, between cognitive group 2 and group 5, 6, there is a significant increase in use of counting strategies from 12% to 33% ($\chi^2 = 10.40$, $\underline{p} <$ .01) and a corresponding decrease in use of inappropriate strategies from 29% to 2% ($\chi^2 = 30.86$, $\underline{p} < .01$). Unexpectedly, other strategies are used at about the same frequency by children at all cognitive levels.

The data for the E problems, also given only to third graders, are summarized in Table 24. As for the D problems, from group 2 to group 5, 6, use of counting strategies increased significantly from 4% to 32% ($\chi^2 = 20.50$, $\underline{p} < .01$) and use of inappropriate strategies decreased from 44% to 5% ($\chi^2 = 46.52$, $\underline{p} < .01$). For both Level D tasks and Level E tasks, there was no appreciable increase in use of algorithms by children at higher cognitive groups (D, 21% to 25%; E, 26% to 25%).

Table 23

Frequency of Use of Strategies by Cognitive Group and Category for All D Tasks

Cognitive Group	Responses	Direct Modeling		Counting Sequences		Routine Mental Operation		Nonroutine Mental Operation		Inappropriate		Algorithm		Non-Algorithm		Incorrect Sentence	
		Frequency	Percent	Frequency	Percent	Frequency	Percent	Frequency	Percent	Frequency	Percent	Frequency	Percent	Frequency	Percent	Frequency	Percent
2	72	14	19	9	12	8	11	3	4	21	29	15	21	1	1	1	1
3	132	29	22	27	20	24	18	5	4	21	16	25	19	0	0	1	1
4	127	25	20	28	22	23	18	9	7	15	12	24	19	1	1	2	2
5,6	126	20	16	42	33	27	21	1	1	3	2	31	25	2	2	0	0

Table 24

Frequency of Use of Strategies by Cognitive Group and Category for All E Tasks

Cognitive Group	Responses	Direct Modeling		Counting Sequences		Routine Mental Operation		Nonroutine Mental Operation		Inappropriate		Algorithm		Non-Algorithm		Incorrect Sentence	
		Frequency	Percent	Frequency	Percent	Frequency	Percent	Frequency	Percent	Frequency	Percent	Frequency	Percent	Frequency	Percent	Frequency	Percent
2	72	5	7	3	4	7	10	2	3	32	44	19	26	2	3	2	1
3	132	28	21	30	23	17	13	0	0	30	23	24	18	1	1	2	1
4	127	29	23	17	13	17	13	1	1	32	25	27	21	0	0	2	2
5,6	126	28	22	40	32	16	13	1	1	6	5	31	25	1	1	3	2

Tasks by Cognitive Group

Within each problem set, one item representing each of six tasks (change/join; change/separate; combine/part unknown; combine/whole unknown; compare, and change join/change/set unknown) was given. From past research (e.g., Carpenter & Moser, 1982), we anticipated that different strategies would be used on tasks with differing semantic structures (particularly on the missing addend problems, Tasks 3 and 6, and on the compare problem, Task 5). The strategy data for each cognitive group for each task for the B set of problems are presented in Table 25. A consistent inverse relationship between use of inappropriate strategies and cognitive level is apparent. Although the percentages of various strategies used with each of the tasks differs, the patterns of use seem to be consistent across cognitive groups. For example, direct modeling is not used by very many students for the compare and change/join missing addend tasks (Tasks 5 and 6) regardless of cognitive group. In particular, counting sequences are used most frequently with Task 6.

The strategy data for each cognitive group on each task for the C set of problems are presented in Table 26. Again, the use of direct modeling goes down with higher cognitive group as does use of inappropriate strategies while use of counting sequences and routine mental operations in general increase. Cognitive group 1 children directly model or use inappropriate strategies across all tasks. The use of other strategies varies by task. Again, direct modeling is not used often with Tasks 5 and 6.

The same data for the D and E sets of problems are shown in Table 27, and again the same pattern is evident. Direct modeling strategies

Table 25

Frequency of Use of Strategies by Cognitive Group and Category for Each B Task

Cognitive Group	N	Direct Modeling		Counting Sequences		Routine Mental Operation		Nonroutine Mental Operation		Inappropriate		
		Frequency	Percent	Frequency	Percent	Frequency	Percent	Frequency	Percent	Frequency	Percent	Trials
Task 1 Change/Join (+)												
1	15	16	53	3	10	3	10	4	13	4	13	30
2	26	23	44	5	10	17	33	3	6	4	8	52
3	9	6	33	3	17	7	39	2	11	0	0	18
Total	50	45	45	11	11	27	27	9	9	8	8	100
Task 2 Change/Separate (-)												
1	15	17	57	1	3	2	7	0	0	10	33	30
2	26	27	52	3	6	16	31	2	4	4	8	52
3	9	10	56	1	6	6	33	0	0	1	6	18
Total	50	54	54	5	5	24	24	2	2	15	15	100
Task 3 Combine/Part Unknown (-)												
1	15	12	40	1	3	2	7	0	0	15	50	30
2	26	26	50	1	2	14	27	1	2	10	19	52
3	9	9	50	2	11	5	28	2	11	0	0	18
Total	50	47	47	4	4	21	21	3	3	25	25	100

(continued)

Table 25 (continued)

Cognitive Group	N	Direct Modeling		Counting Sequence		Routine Mental Operation		Nonroutine Mental Operation		Inappropriate		Trials
		Frequency	Percent	Frequency	Percent	Frequency	Percent	Frequency	Percent	Frequency	Percent	
Task 4 Combine/Whole Unknown (+)												
1	15	17	57	3	10	3	10	1	3	6	20	30
2	26	25	48	8	15	14	27	0	0	5	10	52
3	9	10	56	2	11	6	33	0	0	0	0	18
Total	50	52	52	13	13	23	23	1	1	11	11	100
Task 5 Compare (-)												
1	15	2	7	1	3	2	7	0	0	25	83	30
2	26	10	19	7	13	4	8	1	2	30	58	52
3	9	3	17	1	6	8	44	0	0	6	33	18
Total	50	15	15	9	9	14	14	1	1	61	61	100
Task 6 Change/Join, Change set unknown (-)												
1	15	5	17	11	37	3	10	0	0	11	37	30
2	26	9	17	19	36	20	38	1	2	3	6	52
3	9	1	6	8	44	7	39	1	6	1	6	18
Total	50	15	15	38	38	30	30	2	2	15	15	100

Table 26

Frequency of Use of Strategies by Cognitive Group and Category for Each C Task

Cognitive Group	N	Direct Modeling		Counting Sequence		Routine Mental Operation		Nonroutine Mental Operation		Inappropriate		Trials
		Frequency	Percent	Frequency	Percent	Frequency	Percent	Frequency	Percent	Frequency	Percent	
Task 1 Change/Join (+)												
1	15	11	37	0	0	0	0	0	0	19	63	30
2	38	32	42	13	17	15	20	6	8	10	13	76
3	33	12	18	23	35	19	29	7	11	5	8	66
4	22	1	2	15	34	18	41	7	16	3	7	44
5,6	21	1	2	14	33	21	50	6	14	0	0	42
Total	129	57	22	65	25	73	28	26	10	37	14	258
Task 2 Change/Separate (-)												
1	15	12	40	0	0	0	0	0	0	18	60	30
2	38	33	43	5	6	12	16	5	6	21	28	76
3	33	15	23	17	26	12	18	12	18	10	15	66
4	22	5	11	17	39	12	27	5	11	5	11	44
5,6	21	8	19	20	48	10	24	4	9	0	0	42
Total	129	73	28	59	23	46	18	26	10	54	21	258
Task 3 Combine/Part Unknown (-)												
1	15	11	37	0	0	1	3	0	0	18	60	30
2	38	32	42	14	18	8	10	5	6	17	22	76
3	33	19	27	17	26	17	26	8	12	5	8	66
4	22	4	9	14	32	11	25	8	18	7	16	44
5,6	21	7	17	13	31	20	48	2	5	0	0	42
Total	129	73	28	58	22	57	22	23	9	47	18	258

(continued)

Table 26 (continued)

Cognitive Group	N	Direct Modeling		Counting Sequences		Routine Mental Operation		Nonroutine Mental Operation		Inappropriate		Trials
		Frequency	Percent	Frequency	Percent	Frequency	Percent	Frequency	Percent	Frequency	Percent	
Task 4 Combine/Whole Unknown (+)												
1	15	11	37	0	0	0	0	0	0	19	63	30
2	38	34	45	18	24	10	13	0	0	14	18	76
3	33	9	14	28	42	17	26	4	6	8	12	66
4	22	7	16	11	25	17	39	7	16	2	4	44
5,6	21	5	12	16	38	20	48	1	2	0	0	42
Total	129	66	26	73	28	64	25	12	5	43	17	258
Task 5 Compare (-)												
1	15	1	3	0	0	0	0	0	0	29	97	30
2	38	14	18	12	16	4	5	6	8	40	53	76
3	33	9	14	24	36	16	24	2	3	15	23	66
4	22	9	20	12	27	14	32	5	11	4	9	44
5,6	21	7	17	20	48	15	36	0	0	0	0	42
Total	129	40	16	68	26	49	19	13	5	88	34	258
Task 6 Change/Join, Change set unknown (-)												
1	15	4	13	1	3	1	3	0	0	24	80	30
2	38	21	28	20	26	10	13	5	6	20	26	76
3	33	7	11	21	32	23	35	5	8	10	15	66
4	22	4	9	10	23	20	45	6	14	4	9	44
5,6	21	4	10	18	43	19	45	1	2	0	0	42
Total	129	40	16	70	27	73	28	17	6	58	22	258

Table 27

Frequency of Use of Strategies by Cognitive Group and Category for Each D,E Task

		No Sentence										Correct Sentence				Incorrect Sentence		
		Direct Modeling		Counting Sequences		Routine Mental Operation		Nonroutine Mental Operation		Inappropriate		Algorithm		Non-Algorithm		All Strategies		
Cognitive Group	N	Frequency	Percent	Frequency	Percent	Frequency	Percent	Frequency	Percent	Frequency	Percent	Frequency	Percent	Frequency	Percent	Frequency	Percent	Trials
								Task 1 Change/Join (+)										
2	12	1	4	2	8	2	8	0	0	6	25	12	50	1	4	0	0	24
3	22	7	16	6	14	15	34	1	2	2	4	12	27	1	2	0	0	44
4	21	2	5	2	5	16	38	1	2	5	12	16	38	0	0	0	0	42
5,6	21	4	10	3	7	16	38	0	0	0	0	19	45	0	0	0	0	42
Total	76	14	9	13	9	49	32	2	1	13	9	59	39	2	1	0	0	152
								Task 2 Change/Separate (-)										
2	12	4	17	0	0	2	8	0	0	8	33	8	33	1	4	1	4	24
3	22	15	34	7	16	2	4	1	4	6	13	13	29	0	0	0	0	44
4	21	13	31	6	14	6	14	0	0	4	10	11	26	1	2	1	2	42
5,6	21	13	31	5	12	2	5	0	0	2	5	16	38	2	5	2	5	42
Total	76	45	30	18	12	12	8	1	0	20	13	48	31	4	3	4	3	152
								Task 3 Combine/Part Unknown (-)										
2	12	4	17	2	8	3	12	1	4	10	41	4	17	0	0	0	0	24
3	22	13	29	10	23	4	9	0	0	11	25	5	11	0	0	1	2	44
4	21	13	31	7	17	3	7	1	2	13	31	4	9	0	0	1	2	42
5,6	21	14	33	17	40	5	12	0	0	2	5	3	7	1	2	0	0	42
Total	76	44	29	36	24	15	10	2	1	36	23	16	10	1	1	2	1	152

(continued)

Table 27 (continued)

		No Sentence										Correct Sentence				Incorrect Sentence		
		Direct Modeling		Counting Sequences		Routine Mental Operation		Nonroutine Mental Operation		Inappropriate		Algorithm		Non-Algorithm		All Strategies		
Cognitive Group	N	Frequency	Percent	Frequency	Percent	Frequency	Percent	Frequency	Percent	Frequency	Percent	Frequency	Percent	Frequency	Percent	Frequency	Percent	Trials
Task 4 Combine/Whole Unknown (+)																		
2	12	1	4	1	4	4	17	0	0	10	42	8	33	0	0	0	0	24
3	22	7	16	2	4	12	27	0	0	10	23	13	29	0	0	0	0	44
4	21	5	12	4	9	8	19	0	0	6	14	19	45	0	0	0	0	42
5,6	21	4	9	6	14	11	26	0	0	0	0	21	50	0	0	0	0	42
Total	76	17	11	13	9	35	23	0	0	26	17	61	40	0	0	0	0	152
Task 5 Compare (-)																		
2	12	4	17	4	17	2	8	0	0	12	50	2	8	0	0	0	0	24
3	22	8	18	16	36	3	7	1	2	11	25	3	7	0	0	2	4	44
4	21	9	21	13	31	3	7	5	12	10	24	0	0	0	0	2	5	42
5,6	21	6	14	25	60	3	7	1	2	4	9	2	5	0	0	1	2	42
Total	76	27	18	58	38	11	7	7	5	37	24	7	5	0	0	5	3	152
Task 6 Change/Join, Change set unknown (-)																		
2	12	5	21	3	12	2	8	4	17	7	29	0	0	1	4	2	8	24
3	22	7	16	16	36	5	11	2	4	11	25	3	7	0	0	0	0	44
4	21	12	29	13	31	4	9	3	7	9	21	1	2	0	0	0	0	42
5,6	21	7	17	26	62	6	14	1	2	1	2	1	2	0	0	0	0	42
Total	76	31	20	58	38	17	11	10	7	28	18	5	3	1	1	2	1	152

are now used only for subtraction tasks. Routine mental operations or algorithms are used on Tasks 1 and 4 (the addition tasks) and algorithms on Task 2 (the simplest subtraction task). Direct modeling is often used on all subtraction tasks but rarely for addition, and counting sequences are often used with the missing addend problems (Tasks 3 and 6) and the compare problem (Task 5). A considerable increase in use of counting sequences is apparent from cognitive group 2 to cognitive group 5, 6 on these tasks. However, little difference is seen between group 3 and group 4 children in use of these strategies by grades; there is still a significant difference between use of inappropriate strategies and cognitive group for the students.

In summary, there are important variations in strategies used due to problem set (size of number) and due to specific tasks. Yet, what is clear from this data is that there are important interactions between children who have been identified as having different cognitive processing capabilities and problem set and task. Different strategies on these addition and subtraction tasks regardless of the other important factors are used by children with different capacities.

Chapter 4

COGNITIVE PROCESSING CAPACITY AND CHILDREN'S PERFORMANCE ON STANDARD ADDITION AND SUBTRACTION PROBLEMS

In this chapter, the fourth study in this set is reported. Its purpose was to relate the children's cognitive capacity and grade level to their performance on a standard set of items related to addition and subtraction. The strategy used in this study was achievement monitoring (Romberg & Braswell, 1973). This procedure involves repeatedly measuring groups of students in a quasi-experimental design (Campbell & Stanley, 1963). The measures were objective referenced sets of items on various aspects of addition and subtraction. The quasi-experimental design involved combining longitudinal and cross-sectional designs. In Figure 2, the procedure for describing both the longitudinal and cross-sectional data is shown. The within grade longitudinal growth is represented by the relative heights of the unshaded planes for the group for students in each grade. The shaded plane cross grades parallel to the time of testing axis is the cross sectional growth representation.

The data gathered in this study are summarized first in terms of percent correct on the scales for each grade to portray longitudinal growth. Second, cross sectional growth profiles are presented on the common scales across grades. Third, summarizations of performance are made for students belonging to the same cognitive groups by grade and across grades. Then in conclusion, we related these data for third

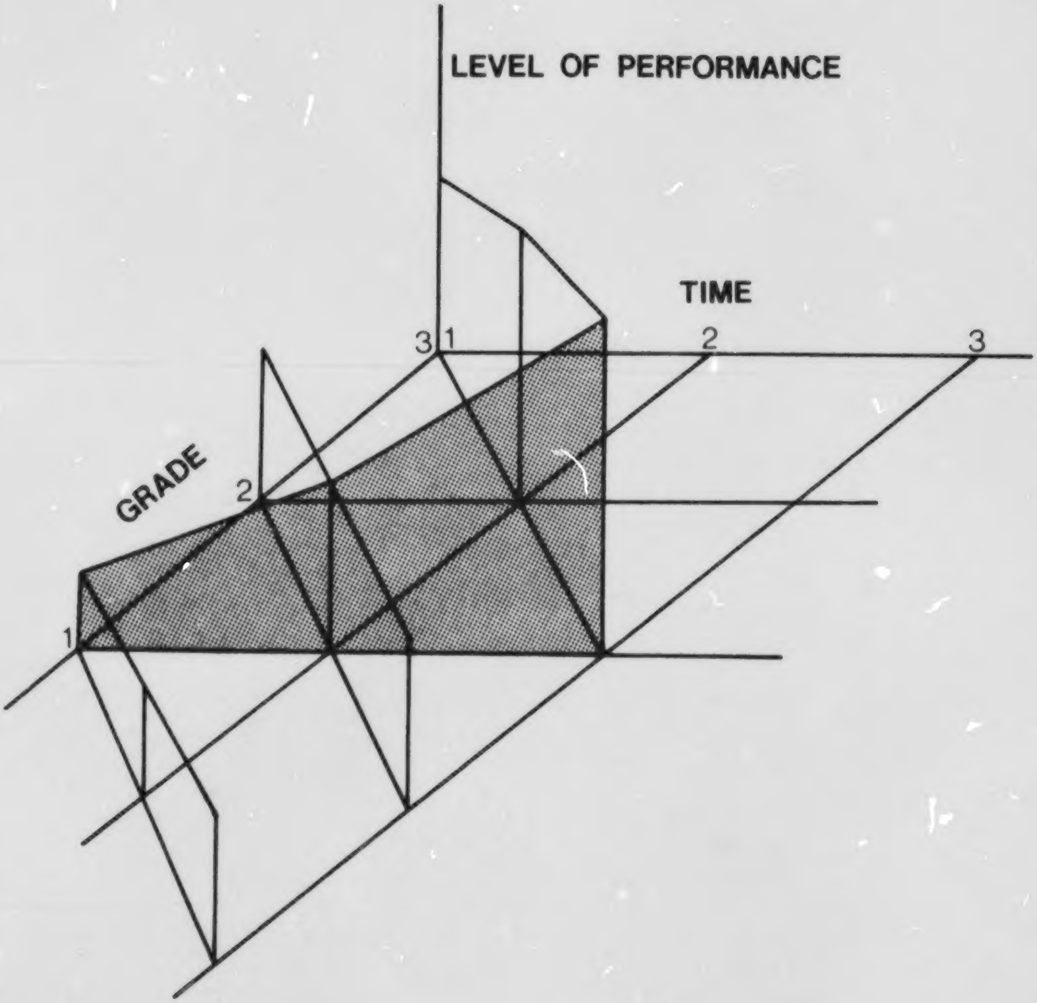
LEVEL OF PERFORMANCE
TIME
GRADE
3
1
2
3
2
1

Figure 2. Longitudinal mean growth (unshaded planes) and crosssectional growth (shaded plane) for grades 1, 2, and 3 students.

grade children to the strategies they used to solve the verbal problems (D and E problems) discussed in Chapter 3.

A sample of children in each grade from the population examined in the previous studies in this series (see Chapters 2 and 3) were administered a set of items on three occasions over a three- to four-month period in 1980 (February 29, April 11, and May 28 or July 6). In each administration, a set of test items was given to each student. Each child's performance on all items was scored. This report presents the data from those test administrations.

Description of the Tests

A battery of paper-and-pencil objective referenced tests had previously been developed to monitor student achievement on addition and subtraction skills at grades 1, 2, and 3 (Buchanan & Romberg, 1983). The battery contained three test forms for each grade. The items were written to assess the instructional objectives of ten experimental topics designed to teach addition and subtraction as well as to measure performance on certain prerequisite objectives and noninstructional objectives (Romberg, Carpenter, & Moser, 1978). A summary of all objectives included in the battery is provided in Table 29. Not all objectives were assessed at all grade levels, however. For this study, because of the small sample of students to be tested, one of the three forms was administered at each grade (Form K at Grade 1, Form S at Grade 2, Form V at Grade 3).

Form K was a 30-minute test containing three subtests: a 15-item multiple-choice subtest and two separate 9-item subtests assessing recall of addition and subtraction facts under speeded conditions. Form S was a 35-minute test containing four subtests; three of the

Table 28

Children at each Cognitive Group in each Grade

	Sandy Bay Infant School		Waimea Heights Primary School	
Cognitive Group	Grade 1	Grade 2	Grade 3	Total
1	3	2	0	5
2	3	6	4	13
3	1	2	8	11
4	0	0	8	8
5,6	0	0	7	7
Total	7	10	27	44

Table 29

Objectives Assessed in Addition and Subtraction

Achievement Monitoring Battery

Prerequisite Instructional Objectives

- Numerousness
  - 0-10
  - 11-20
  - 0-99, writes
  - 0-99, represents
- Ordering, Place Value
  - sets, one-to-one correspondence
  - numbers 0-20
  - numbers 0-99, orders
  - numbers 0-99, notation

Instructional Objectives for the S and A Topic Series

- Open Sentences
  - add 0-20
  - subt 0-20
- Sentence-Writing 0-20
  - add-simple joining
  - subt-simple separating
  - subt-part part whole-addend
  - add- part part whole
  - subt-comparison
  - subt-join-addend
- Sentence-Writing 0-99
  - add-simple joining
  - subt-simple separating
  - subt-part part whole-addend
  - add-part part whole
  - subt-comparison
  - subt-join-addend
- Algorithms
  - add 0-99
  - subt 0-99

Non-instructional Objectives

- Problem-Solving 0-20
  - add-simple joining
  - subt-simple separating
  - subt-part part whole-addend
  - add-part part whole
  - subt-comparison
  - subt-join-addend
- Problem-Solving 0-99
  - add-simple joining
  - subt-simple separating
  - subt-part part whole-addend
  - add-part part whole
  - subt-comparison
  - subt-join-addend
- Counting 9-31
  - on
  - back
- Basic Facts--Speeded Test
  - add 0-20
  - subt 0-20
- Algorithms--Timed Test
  - add 0-99
  - add 0-99

---

subtests were similar to the Form K subtests with some items dropped and some added forming a 19-item multiple-choice subtest and two 12-item recall tests. The fourth subtest was a 4-item free response sentence-writing measure. Form V for third grade was a 40-minute test containing six subtests. In this case, the two recall subtests and the sentence-writing subtest were identical for the Form S subtests. Five items were dropped from the Form S multiple-choice subtest leaving 14 items. The two new subtests were 24-item timed measures of performance on addition and subtraction algorithms.

Multiple-choice subtests. Individual objectives in the areas of numerousness, ordering, place value, open sentences, and algorithms were represented by one multiple-choice item in each test form on which they were assessed. For the two objectives for counting, counting on and counting back for numbers to 18, there was one item per form; however, an additional counting item for numbers to 31 was included in each test because information on these numbers was of potential interest relative to interview problem situations using larger items (see Chapter 3).

Four individual objectives for sentence-writing were represented by a multiple-choice item in each form. For grade 1, these items contained numbers 5-9 or 11-15; for grades 2 and 3 the number domains were 11-15 and 0-99. Since there was no way in a multiple-choice format to have students actually write a sentence, the items required listening to a verbal problem read aloud and then choosing the sentence which correctly represented the verbal situation. The problem situation itself was not printed on the test page. This prevented reading difficulties and also was in keeping with the procedures for the interviews in which the problems were presented orally.

For Form K, two objectives for the problem-solving area were assessed while for Forms S and V, four objectives were included. The number domains were the same as for the sentence-writing objectives, and again, the problem situations were not printed in the student booklets.

All of the questions in the multiple-choice section of the tests were read to the children and then the key phrases were repeated; in the case of the verbal problems for the sentence-writing and problem-solving objectives, the entire story situation was read twice. The children then marked an X on one of the four response choices: the solution, two distractors, and the "puzzled face," an option which indicated "I have not learned this yet." The response choices, symbols, and pictures were not read or explained to the children (with the exception of the "puzzled face").

The "puzzled face" option was provided to avoid unnecessary frustration and to reduce the amount of random guessing. Although it was expected that the "puzzled face" choice would be used throughout the achievement testing because there would always be objectives not yet introduced and/or mastered, this option was particularly useful at the baseline period. Marking the "puzzled face" allowed children to give a positive response indicating that they hadn't yet learned to find the answer to the question.

Speeded subtests. There were 9 addition and 9 subtraction facts on Form K and 12 on each of Forms S and V. The first six problems in each case covered the facts from 4 to 9; the last three (or six) involved 10 to 18. The addition and subtraction recall subtests were introduced by the test administrator; then specific directions on a tape recording preceded the items presented with intervals of 4 seconds working time for Form K and 2 seconds for Forms S and V. The children wrote their answers in designated spaces, leaving spaces for unknown facts empty. There was a short break between the two subtests.

Sentence-writing free response subtests. Four of the 12 individual sentence-writing objectives (verbal problem types) for the numbers 0-20 and 0-99 were assessed in Forms S and V. A free response format was employed in which a verbal problem was read twice to the students who were directed to write a sentence for the situation and not solve the sentence. There were two 0-20 and two 0-99 items per test.

Addition and subtraction algorithms timed subtests. These subtests, in Form V only, each contained 24 items. The items were either 2-digit or 3-digit; 18 items required regrouping, 6 did not. The items were arranged in order of difficulty. For example, 3-digit problems not requiring regrouping preceded 3-digit problems which required regrouping and, for 3-digit regrouping problems, those in which only the ones were regrouped preceded those in which both ones and tens were regrouped. The students were instructed to try each problem in order (the problems were alphabetized) and to go on to the next problem if unable to do a particular example. Six minutes was allowed for each subtest.

Test Administration

The three assistants who gathered data in Study 3 (Chapter 3) also carried out that task in this study. Guidelines for administering the achievement tests were provided to each assistant. The guidelines indicated which tests were to be given, dates for administration, and so forth.

The first administration was supervised by Professor Romberg and went smoothly. The second and third administrations were carried out after Professor Romberg had returned to the U.S. These test administrations at grade 1 went smoothly as scheduled. At grade 2 one item on Form S did not copy well so students could not read that question. At grade 3 there were two administrative mixups. First, Form S rather than Form V was given in April to all three classes and in May to two of the classes. This was not a serious problem since many items are the same, except that the timed algorithms tests were not given. Second, in the third class Form V was given in July rather than May. The May administration was scheduled for near the end of the autumn term, but the assistant failed to administer the tests at that time. After a short break, children returned to school to start the winter term. The assistant asked whether she should still gather the data and was advised to administer Form V in July. The results of this administration would not reflect much additional instruction since there had been a break between terms. All data were then shipped to Madison and scored by Center staff. Each subject's responses were recorded and are the basis for all summary information appearing in this paper.

Longitudinal Growth Within Grades

Grade 1. The percent correct for students at grade 1 on the individual objectives and composite objectives for each of the three administrations is shown in Table 30. Overall, the data show that this sample of students at the start of the school year (February) had acquired the prerequisite objectives and could solve the verbal addition problems (but probably not by addition), and some (43%) could find the answer to an open addition problem. They could not solve subtraction problems, write sentences, count on or count back, nor could they recall basic facts.

By the end of the autumn term (May), the addition skills of these students had improved dramatically. The percent correct improved for solving an open sentence, 43% to 86%; writing a correct addition sentence, 29% to 57%; counting on, 29% to 57%; and addition facts, 33% to 76%. However, the same cannot be said for subtraction. Only for solving a verbal comparison problem (29% to 71%) and for subtraction facts (29% to 56%) was there marked improvement. Obviously, instruction in grade 1 had some effect.

Grade 2. For grade 2 students, the picture is somewhat different (see Table 31). Overall for this sample of nine students, at the beginning of the school year the percent correct was quite low. In fact, on only three items did more than half of the students get the correct answer. Part of the difficulty was that Form S used large numbers (0-99) in several of the questions. By May, improvement on several composite objectives was apparent. The students were comfortable with numerousness of larger sets (56% to 75%), had improved on basic facts (29% to 51% and 23% to 53%, but not yet to any

Table 30

Percent Correct for Objectives and Composite Objectives by

Administration Time for Grade 1, Form K

Description of Objectives	Results for Objectives				Results for Composite Objectives			
	Number of Items	Feb. N=7	April N=7	May N=7	Number of Items	Feb. N=7	April N=7	May N=7
Prerequisite Instructional Objectives								
Numerousness								
0-10	1	100	100	100	2	86	71	93
11-20	1	71	43	86				
Ordering								
sets, one-to-one correspondence	1	86	71	86	2	93	86	86
numbers 0-20	1	100	100	86				
Instructional Objectives for S Topics								
Open Sentences								
add 0-20	1	43	57	86	2	29	36	50
subt 0-20	1	14	14	14				
Sentence-Writing 0-20								
subt-simple separating (11-15)	1	14	0	0	4	21	11	21
subt-comparison (5-9)	1	29	14	0				
add-simple joining (11-15)	1	29	14	57				
subt-part part whole-addend (11-15)	1	14	14	29				
Noninstructional Objectives								
Problem Solving 0-20								
add-part part whole (5-9)	1	100	100	100	2	64	57	86
subt-comparison (11-15)	1	29	14	71				
Counting On 9-31	2	29	43	57	3	19	33	43
Counting Back 9-31	1	0	14	14				
Recall of Basic Facts--Speeded Test								
add 0-20					9	33	49	76
subt 0-20					9	29	44	56

Table 31

Percent Correct for Objectives and Composite Objectives by Administration Time for Grade 2, Form S

Description of Objectives	Results for Objectives: Number of Items	Feb. N=9	April N=9	May N=8	Results for Composite Objectives: Number of Items	Feb. N=9	April N=9	May N=8
Prerequisite Instructional Objectives								
Numerousness								
writes 0-99	1	--	--	--[a]	1	56	67	75
represents 0-99	1	56	67	75				
Ordering, Place Value								
ordering 0-99	1	11	0	25	2	6	0	19
place value 0-99	1	0	0	13				
Instructional Objectives for S and A Topics								
Open Sentences								
add 0-20	1	22	78	100	2	17	39	88
subt 0-20	1	11	0	75				
Sentence-Writing 0-20, 0-99 (multiple choice)								
subt-simple separating (11-15)	1	33	33	25	4	17	14	16
subt-comparison (0-99)	1	0	0	0				
add-simple joining (0-99)	1	11	11	25				
subt-part part whole-addend (11-15)	1	22	11	13				
Sentence-Writing 0-20, 0-99 (free response)								
subt-simple separating (0-99)	1	56	44	75	4	28	53	59
subt-part part whole-addend (0-99)	1	0	0	0				
add-part part whole (11-15)	1	56	89	100				
subt-join-addend (11-15)	1	0	78	63				
Algorithms								
addition algorithm	1	11	33	13	2	11	17	25
subtraction algorithm	1	11	0	38				

continued

Table 31 (continued)

Description of Objectives	Results for Objectives				Results for Composite Objectives			
	Number of Items	Feb. N=9	April N=9	May N=8	Number of Items	Feb. N=9	April N=9	May N=8
Noninstructional Objectives								
Problem-Solving 0-20, 0-99								
add-part part whole (0-99)	1	0	22	25	4	22	39	25
subt-comparison (11-15)	1	22	56	50				
subt-part part whole-addend (11-15)	1	44	67	13				
subt-join-addend (0-99)	1	22	11	13				
Counting On 9-31	2	33	28	81	3	30	33	63
Counting Back 9-31	1	22	44	25				
Recall of Basic Facts--Speeded Test								
add 0-20					12	29	35	51
subt 0-20					12	23	30	53

---

[a] Students were unable to complete item because tests duplicated poorly.

level of mastery), could solve simple open sentences (17% to 88%), and had improved in counting (30% to 63%) and writing sentences for verbal problems (28% to 59%). Again, instruction had an effect, but increases in performance were not apparent for ordering large numbers, problem solving, selecting written sentences for verbal problems, and algorithms.

Grade 3. For grade 3 students, the picture was more encouraging (see Table 32). In February, their performance was not high (above 80%) except on two items, but by the end of May (or early July) performance on all composite objectives except one was approaching or about 80%. The one exception was the item on place value for numbers 0-99. Sentence writing-selecting skills had improved, but for some subtraction situations (comparison and part-part-whole addend) scores were not yet high.

Performance of the grade 3 students on the timed algorithms test is shown in Table 33. In February when all 22 children were tested, they performed well on the six addition-without-regrouping problems and fair on the three items testing 2-digit subtraction without regrouping. On all others, they did poorly. Part of the difficulty was that because of the timed conditions most did not attempt the last items in the test. Those children who did reach the items did fairly well on the addition regrouping items but had considerable difficulty with the subtraction items requiring regrouping.

Unfortunately, no children were given this test again in April or May and only 12 in July. By then, performance for those students was considerably better. There was still some difficulty with the

Table 32

Percent Correct for Objectives and Composite Objectives by Administration Time for Grade 3, Forms S, V

Description of Objectives	Number of Items	Feb. N=22	April N=22	May/July[a] N=11/12	Results for Composite Objectives: Number of Items	Feb. N=22	April N=22	May/July N=11/12
Prerequisite Instructional Objectives								
Numerousness								
writes 0-99	1	45	32	64/92	2	68	61	82/92
represents 0-99	1	91	91	100/91				
Ordering, Place Value								
ordering 0-99	1	36	91	64/75	2	30	70	32/58
place value 0-99	1	23	50	0/42				
Instructional Objectives for S and A Topics								
Sentence-Writing 0-20, 0-99 (multiple choice)								
subt-simple separating (11-15)	1	60	91	73/100	4	41	61	41/83
subt-comparison (0-99)	1	18	14	18/58				
add-simple joining (0-99)	1	77	91	64/100				
subt-part part whole-addend (11-15)	1	10	50	9/75				
Sentence-Writing 0-20, 0-99 (free response)								
subt-simple separating (0-99)	1	36	77	82/92	4	39	64	64/81
subt-part part whole-addend (0-99)	1	5	23	18/67				
add-part part whole (11-15)	1	68	95	100/92				
subt-join-addend (11-15)	1	45	60	55/75				

continued

Table 32 (continued)

Description of Objectives	Results for Objectives				Results for Composite Objectives			
	Number of Items	Feb. N=22	April N=22	May/July[a] N=11/12	Number of Items	Feb. N=22	April N=22	May/July N=11/12
Noninstructional Objectives								
Problem-Solving 0-20, 0-99								
add-part part whole (0-99)	1	55	68	64/92	4	67	78	80/87
subt-comparison (11-15)	1	91	77	100/100				
subt-part part whole-addend (11-15)	1	77	95	91/83				
subt-join-addend (0-99)	1	45	73	64/75				
Recall of Basic Facts--Speeded Test								
add 0-20					12	44	66	66/94
subt 0-20					12	40	69	52/84
Algorithms--Timed Test								
addition algorithm					24	41	--[b]	--/81[b]
subtraction algorithm					24	15	--[b]	--/65[b]

[a] Form S was used in April and May; Form V was used in February and July.

[b] Form S did not assess this objective.

Table 33

Percent Correct for Addition and Subtraction Algorithms

Timed Tests by Problem Type for Grade 3, Form V

Item Type	Number of Items	Percent Correct Feb. $\underline{N}$=22	 July $\underline{N}$=12
Addition			
2-digit (without regrouping)	3	86	100
3-digit (without regrouping)	3	93	94
2-digit (with regrouping)[a]	6	49	89
3-digit (with regrouping)	9	16	78
3-digit addends	3	0	44
Subtraction			
2-digit (without regrouping)	3	68	94
3-digit (without regrouping)	3	33	89
2-digit (with regrouping)[a]	6	8	75
3-digit (with regrouping)	12	0	47

[a] 3 items are 2-digit ± 1-digit.

three-addend addition problems and the subtraction regrouping problems but the increases in every case are striking.

In summary, for this small sample of children assessed at each grade level, growth within each grade on some aspects associated with addition and subtraction is clear. Growth, however, is not uniform across objectives. In addition, overall level of performance on many objectives is not high. Students by mid-third grade have yet to master many aspects of either addition or subtraction.

The overall picture these data presents is of children struggling to learn the complex arithmetic skills associated with addition and subtraction and to use those skills to solve verbal problems. children had difficulty with place value even though they correctly answered 3-digit problems. Work on algorithms improved even though basic facts were weak. And children correctly solved some simple verbal problems with little arithmetic competence.

Cross-sectional Growth Across Grades

To portray cross sectional growth (see Figure 2), five objectives were assessed in all three grades: sentence writing: subtraction-simple separating (11-15); sentence writing: subtraction-part-part-whole missing addend (11-15); problem solving subtraction-comparison (11-15); recall of basic facts-addition; and recall of basic facts-subtraction. Also two composite scales were administered to both grade 1 and grade 2 children, and the composite scale ordering, place value was administered at both grades 2 and 3.

The cross-sectional data for these scales are presented in Table 34. On each objective, considerable growth is evident. But, as with the longitudinal data, the growth is not uniform or smooth.

Table 34

Percent Correct for Common Objectives and Composite

Objectives for Cross-sectional Growth Across Grades 1, 2, and 3

Description of Objective	Percent Correct Feb. Grade 1 N=7	 April Grade 2 N=9	 May/July[a] Grade 3 N=23
Sentence Writing			
subt-simple separating (11-15)	14	33	87
subt-part part whole-addend (11-15)	14	11	39
Problem Solving			
subt-comparison	29	56	100
Recall of Basic Facts--Speeded Test			
add 0-20	33	35	78
subt 0-20	29	30	69
	Feb. Grade 1 N=7		May Grade 2 N=8
Open Sentences	29		88
Counting On and Back	19		63
	Feb. Grade 2 N=9		May/July[a] Grade 3 N=23
Ordering, Place Value	6		46

[a] Data gathered on these dates have been combined.

Performance of Children in Cognitive Groups Within Grades

Because of the small sample of students to summarize data for children in different cognitive groups, their scores have been aggregated into a total score across all three administrations of the tests.

Grade 1. The relative performance on the test items for children in grade 1 in different cognitive groups is shown in Table 35. There were three children in both cognitive Groups 1 and 2, but only one child in cognitive Group 3. The differences in performance for the eight composite objectives favor the Group 2 children over the Group 1 children on six composites with some of the differences being quite large. In addition, the Group 2 students increased in performance from February to May over all the objectives but the Group 1 students improved only in recall of facts. The single Group 3 child fails to fit any pattern.

Grade 2. For the grade 2 children, the relative performance for children at different cognitive groups is shown in Table 36. There were two children in cognitive Groups 1 and 3 and five in Group 2. In general, the pattern shows Group 3 children performing better than Group 2 children who in turn do better than the Group 1 children. Some of the differences are striking, for example, open sentences (58%-46%-33%) and addition facts (60%-35%-24%). However, there is one anomaly. For the four problem-solving items, the Group 1 children did better than either other group (46% to 20% to 33%). However, since these children were low on facts, algorithms, and counting skills, the results suggest that they found answers to the verbal problems using other strategies. The children with better arithmetic skills (but not

Table 35

Frequency and Percent Correct for Composite Objectives by Cognitive Group for All Administration Times for Grade 1, Form K

Description of Objectives	Number of Items	Cognitive Group 1			Cognitive Group 2			Cognitive Group 3			Total		
		Frequency	Percent	Trials	Frequency	Percent	Trials	Frequency	Percent	Trials	Frequency	Percent	Trials
Prerequisite Instructional Objectives													
Numerousness 0-20	2	14	78	18	17	94	18	4	67	6	35	83	42
Ordering 0-20	2	16	89	18	15	83	18	6	100	6	37	88	42
Instructional Objectives for the S Topics													
Open Sentences	2	7	39	18	7	39	18	2	33	6	16	38	42
Sentence-Writing 0-20	4	4	11	36	9	25	36	2	17	12	15	18	84
Noninstructional Objectives													
Problem Solving 0-20	2	12	67	18	13	72	18	4	67	6	29	69	42
Counting	3	2	7	27	16	59	27	2	22	9	20	32	63
Addition Facts Recall--Speeded Test	9	24	30	81	65	80	81	11	41	27	100	53	189
Subtraction Facts Recall--Speeded Test	9	24	30	81	49	60	81	8	30	27	81	43	189

Table 36

Frequency and Percent Correct for Composite Objectives by Cognitive Group for All Administration Times for Grade 2, Form S

Description of Objectives	Number of Items	Cognitive Group 1			Cognitive Group 2			Cognitive Group 3			Total		
		Frequency	Percent	Trials	Frequency	Percent	Trials	Frequency	Percent	Trials	Frequency	Percent	Trials
Prerequisite Instructional Objectives													
Numerousness 0-99	1[a]	4	67	6	8	58	14	5	84	6	17	67	26
Ordering, Place Value 0-99	2	3	25	12	0	0	28	1	8	12	4	8	52
Instructional Objectives for the S and A Topics													
Open Sentences	2	4	33	12	13	46	28	7	58	12	24	46	52
Sentence-Writing 0-20, 0-99 (multiple choice)	4	1	4	24	11	20	56	4	17	24	16	15	104
Sentence-Writing 0-20, 0-99 (free response)	4	9	38	24	25	45	56	14	58	24	48	46	104
Algorithms	2	1	8	12	4	14	28	4	33	12	9	17	52
Noninstructional Objectives													
Problem Solving 0-20, 0-99	4	11	46	24	11	20	56	8	33	24	30	29	104
Counting	3	6	33	18	14	33	42	12	67	18	32	41	78
Addition Facts Recall--Speeded Test	12	17	24	72	58	35	168	43	60	72	118	38	312
Subtraction Facts Recall--Speeded Test	12	16	22	72	50	30	168	43	60	72	109	35	312

[a] Two items were administered for the numerousness objective; students had difficulty reading one of the items due to poor quality of the test duplication so data for this item were discarded.

close to mastery) may have attempted to use those skills to solve the problems, but made errors. This explanation is further substantiated by the decrease in performance of the Group 1 children on those items as the year progresses and as their arithmetic skills improve.

Grade 3. For the grade 3 students in different cognitive groups, the results are striking but somewhat ambiguous (see Table 37). The Group 5,6 children performed better on all objectives than any other group, and Group 2 children were lower than other groups on all the objectives. But Groups 3 and 4 failed to differ in a consistent manner. Obviously, the differing characteristics of these two groups are not related to differences in performance. Most of the differences between the Group 5,6 and the Group 2 children are large (selecting sentences 65% to 44%, ordering 68% to 33%, subtraction algorithms 51% to 13%, and so forth).

In summary, with one important exception, children who were identified as being in a particular cognitive group performed different than children in other groups within each grade. The one exception was the lack of consistent differences between Groups 3 and 4 at grade 3. Again, it should be noted that Group 3 at grade 3 also failed to differ on the interview tasks (see Chapter 3) and only differed on "transitivity" on the cognitive tasks (see Chapter 2). Overall, however, it is very apparent that children who differ in cognitive processing capacity (Group 1, Group 2, Groups 3 and 4, and Group 5,6) performed differently regardless of specific objectives, instruction over time, or grade.

Table 37

Frequency and Percent Correct for Composite Objectives by Cognitive Group for All Administration Times for Grade 3

Description of Objectives	Number of Items	Cognitive Group 2			Cognitive Group 3			Cognitive Group 4			Cognitive Group 5,6			Total		
		Frequency	Percent	Trials	Frequency	Percent	Trials	Frequency	Percent	Trials	Frequency	Percent	Trials	Frequency	Percent	Trials
Prerequisite Instructional Objectives																
Numerousness 0-99	2	15	63	24	29	73	40	24	67	36	29	85	34	97	72	134
Ordering, Place Value 0-99	2	8	33	24	20	50	40	14	39	36	23	68	34	65	49	134
Instructional Objectives for the S and A Topics																
Sentence-Writing 0-20, 0-99 (multiple choice)	4	21	44	48	42	53	80	39	54	72	44	65	68	146	54	268
Sentence-Writing 0-20, 0-99 (free response)	4	25	52	48	49	61	80	37	51	72	44	65	68	155	58	268
Noninstructional Objectives																
Problem Solving 0-20, 0-99	4	34	71	48	58	73	80	55	76	72	58	85	68	205	76	268
Addition Algorithms--Timed Test[a]	24	30	31	96	134	51	264	111	51	216	172	72	240	447	55	816
Subtraction Algorithms--Timed Test[a]	12	12	13	96	78	30	264	55	25	216	122	51	240	267	33	816
Addition Facts Recall--Speeded Test	12	65	45	144	152	63	240	133	62	216	162	79	204	512	64	804
Subtraction Facts Recall--Speeded Test	12	61	42	144	135	56	240	126	58	216	154	75	204	476	59	804

[a] This objective was assessed in February for 22 students representing all cognitive groups and in May for 12 students in all groups except 2. It was not assessed in April.

Performance of Children in Cognitive Groups Across Grades

Performance data from Tables 35, 36, and 37 for children in cognitive capacity Groups 1, 2, and 3 are compared in this section. (Children in Groups 4 and 5,6 are only in grade 3.)

Group 1. The performance of children in this group at both grades 1 and 2 is shown in Table 38. In general, the performance of these children at both grades is consistent with their capacity. Only for ordering small numbers is their performance adequate. More strikingly, there is little difference in performance between grades. Only on the counting items is there a marked difference (7% to 33%) but still performance is very low.

Group 2. Children in this capacity group are at all three grade levels. The comparative data for these children are presented in Table 39. Performance gains by grade are apparent, but in most cases very modest. For example, performance on solving open sentences goes from 39% to 46% from grade 1 to grade 2 or performance on writing sentences (free response) from 45% to 52% from grade 2 to grade 3. Only for problem solving (0-99) was there a marked gain (20% to 71%). Also, there is a marked decrease in performance from grade 1 to grade 2 on performance on recall of both addition and subtraction facts. The decrease is undoubtedly due to the increased number of facts and decreased time for response over forms. This clearly suggests the high performance at grade 1 was not due to having committed them to memory. Also, it should be noted that at grade 3, these children have not learned to use the addition and subtraction algorithms with any facility. Again, overall, the performance of these students reflects level of capacity more than grade level.

Table 38

Frequency and Percent Correct for Composite Objectives for Cognitive Group 1 for All Administration Times Across Grades

Description of Objectives	Grade 1			Grade 2		
	Frequency	Percent	Trials	Frequency	Percent	Trials
Prerequisite Instructional Objectives						
Numerousness 0-20	14	78	18	--	--	--
Numerousness 0-99	--	--	--	4	67	6
Ordering 0-20	16	89	18	--	--	--
Ordering, Place Value 0-99	--	--	--	3	25	12
Instructional Objectives for the S and A Topics						
Open Sentences	7	39	18	4	33	12
Sentence-Writing 0-20	4	11	36	--	--	--
Sentence-Writing 0-20, 0-99 (multiple choice)	--	--	--	1	4	24
Sentence-Writing 0-20, 0-99 (free response)	--	--	--	9	39	24
Algorithms	--	--	--	1	8	12
Noninstructional Objectives						
Problem Solving 0-20	12	67	18	--	--	--
Problem Solving 0-20, 0-99	--	--	--	11	46	24
Counting	2	7	27	6	33	18
Addition Facts Recall--Speeded Test	24	30	81	17	24	72
Subtraction Facts Recall--Speeded Test	24	30	81	16	22	72

Table 39

Frequency and Percent Correct for Composite Objectives for Cognitive Group 2 for All Administration Times Across Grades

Description of Objectives	Grade 1			Grade 2			Grade 3		
	Frequency	Percent	Trials	Frequency	Percent	Trials	Frequency	Percent	Trials
Prerequisite Instructional Objectives									
Numerousness 0-20	17	94	18	--	--	--	--	--	--
Numerousness 0-99	--	--	--	8	58	14	15	63	24
Ordering 0-20	15	83	18	--	--	--	--	--	--
Ordering, Place Value 0-99	--	--	--	0	0	28	8	33	24
Instructional Objectives for the S and A Topics									
Open Sentences	7	39	18	13	46	28	--	--	--
Sentence-Writing 0-20	9	25	36	--	--	--	--	--	--
Sentence-Writing 0-20, 0-99 (multiple choice)	--	--	--	11	20	56	21	44	48
Sentence-Writing 0-20, 0-99 (free response)	--	--	--	25	45	56	25	52	48
Algorithms	--	--	--	4	14	28	--	--	--
Noninstructional Objectives									
Problem Solving 0-20	13	72	18	--	--	--	--	--	--
Problem Solving 0-20, 0-99	--	--	--	11	20	56	34	71	48
Counting	16	59	27	14	33	42	--	--	--
Addition Facts Recall--Speeded Test	65	80	81	58	35	168	65	45	144
Subtraction Facts Recall--Speeded Test	49	60	81	50	30	168	61	42	144
Addition Algorithms	--	--	--	--	--	--	30	31	96
Subtraction Algorithms	--	--	--	--	--	--	12	13	96

Group 3. The data for children in this cognitive capacity group at grades 2 and 3 are compared in Table 40. There are some important differences in performance which are due to grade. For example, performance increases from 8% to 50% on ordering, place value; from 17% to 53% on sentence-writing (multiple choice); and from 33% to 73% on problem solving. However, for all other scales, performance is similar across grades. Overall performance is fair, some facility with the addition algorithm is apparent, but not with the subtraction algorithm.

In summary, while it cannot be denied that teaching or experience accounts for some differences in the level of performance for these children on standard addition and subtraction tasks, what is striking is that the actual level of performance appears to be consistent with capacity. Differences in performance between groups and within groups across grades are differences one could expect based on the nature of the groups (e.g., level of quantitative skills, memory capacity, and so on).

Relationship of Performance on Algorithms to Strategies Used to Solve Problems

One overall goal of instruction on addition and subtraction is that students, when faced with a verbal problem (such as those presented in Chapter 3), would solve those problems using an addition or subtraction algorithm. For the third-grade children in this study, relationship of their performance on the timed algorithm problems to the strategies they used to solve verbal problems which could be done using those algorithms is now examined. The strategy data were collected in the interview study discussed in Chapter 3. We were

Table 40

Frequency and Percent Correct for Composite Objectives for Cognitive Group 3 for All Administration Times Across Grades

Description of Objectives	Grade 2			Grade 3		
	Frequency	Percent	Trials	Frequency	Percent	Trials
Prerequisite Instructional Objectives						
Numerousness	5	84	6	29	73	40
Ordering, Place Value 0-99	7	8	12	20	50	40
Instructional Objectives for the S and A Topics						
Open Sentences	7	58	12	--	--	--
Sentence-Writing 0-20, 0-99 (multiple choice)	4	17	24	42	53	80
Sentence-Writing 0-20, 0-99 (free response)	14	58	24	49	61	80
Algorithms	4	33	12	--	--	--
Noninstructional Objectives						
Problem Solving 0-20	8	33	24	58	73	80
Counting	12	67	18	--	--	--
Addition Facts Recall--Speeded Test	43	60	72	152	63	240
Subtraction Facts Recall--Speeded Test	43	60	72	135	56	240
Addition Algorithms	--	--	--	134	51	264
Subtraction Algorithms	--	--	--	78	30	264

NOTE: The one Group 3 child at grade 1 was not included in this comparison.

particularly interested in examining whether or not students who had learned to use the addition and subtraction algorithms in fact chose to use them when solving such verbal problems.

For addition problems requiring no regrouping, at time 1, 62 items were attempted and 57 were correct (92%) and in July, all 36 items attempted were correct. With one exception (student 517), these students knew how to add 2-digit numbers without regrouping. However, on the interviews at time 1, algorithms were used only 59% of the time (54% correctly). On interviews 2 and 3, the percent of use increased but only to 79% and 72%.

Similar data for addition with regrouping showed at time 1 students attempted 95 items and got 66 correct (69%) and by time 2 they attempted 74 items getting 66 correct (89%). Thus, while there was some difficulty with regrouping at the start of the year, by July, with the exception of one student who made six errors in six problems, the students all could add with regrouping.

The interview data show that in spite of this level of performance, many students did not use the algorithms to solve verbal addition problems. On the interview 1 tasks, about half (54%) of the children tried using an algorithm (46% correctly). On the second interview, this had changed to 60% using an algorithm (48% correctly) and by interview 3, 78% used an algorithm with no errors.

For subtraction without regrouping, performance on three achievement items contrasted with strategies used on the four verbal subtraction problems showed similar results. At time 1, 55 items had been attempted with 45 being correct (82%) and by time 2, 34 of 36 attempts were correct (94%). In fact, only one student made any errors in July. One can conclude that these students were able to subtract without regrouping. However, on the four verbal subtraction problems only 14% of the strategies used were algorithmic (only 9% correct) at the start of the year. By the second interview, this had increased to 25% and finally to 34% by the third interview. Furthermore, over half of the total attempts (59%) were just on Task 2 (simple separate), the most obvious subtraction problem.

The same pattern, only more pronounced, occurred for subtraction with regrouping. At the start of the year, only 38 items were attempted and only 12 were correct (32%). Many children managed only to complete the first six no-regrouping items in this timed test so that there was no real measure of their capability. However, it is hard to imagine why they were so slow, one can only assume that they could have been unable to do the regrouping problems had they attempted them. By the second administration (July), 66 items were attempted and 5 were correct (82%). Also, only two students made more than one error on the six problems. Thus, while there was evidence of considerable difficulty in subtracting with regrouping in February, by the end of the Autumn term, most were capable of using a subtraction algorithm.

But again, in spite of knowing the algorithmic procedures for subtraction, most children did not attempt to solve verbal problems using them. On the first interview, algorithms were used on only 13% of the items (5% correctly). On the second interview, this had increased to 23% (11% correctly), and by the third interview, it was 35% (26% correctly). And, as with subtraction no-regrouping, most of the attempts were on the simple separating tasks (44%).

Also, on this last set of verbal problems, the cognitive Group 2 students make the most total attempts to use algorithms (35% of the time). This is true even though they got no items correct on the achievement test and made the most errors (only 10% correct) on these verbal problems. In contrast, the Group 5,6 students attempted to use algorithms only 22% of the time.

Overall, this relationship between skill of doing addition and subtraction algorithms and using the algorithms to solve verbal problems is interesting. Most third-grade students use other strategies (counting, fingers, and so forth) until they become really confident in using the algorithms. However, Group 2 children who have not acquired other strategies to solve these problems tend to use the taught algorithms even though they are not proficient in their use. This suggests that most students at grade 3 recognize that these problems can be solved using algorithms but chose to use other familiar strategies. The problem structures (verbal semantics) clearly influence how the problems are worked. In fact, the semantics seem to be more important than the realization that the problems could be done algorithmically.

Chapter 5

COGNITIVE PROCESSING CAPACITY AND CLASSROOM INSTRUCTION

The fifth and last study in this series is reported in this chapter. Its purpose was to examine the question: Do children who differ in cognitive capacity receive different instruction?

The Sample for this Study

To examine this question, a sample of children from the population used in studies 3 and 4 in this series (see Chapter 2) were observed during instruction over a three month period in 1980 (February 27 through May 28). The number of children selected to be observed in each cognitive group in each class in each grade is shown in Table 41.

The observational data were gathered from a content perspective. Our attempt was to determine the way in which aspects of content influence certain teacher behaviors during instruction and in turn how these actions affect pupil outcomes. In particular, the extent to which children are engaged in learning mathematics is being examined. To do this a model of classroom instruction was constructed where "content segmentation and sequencing" and "content structuring" were hypothesized to influence teacher planning which in turn influences classroom organization, the allocation of instructional time, verbal interactions within classroom, and, eventually, pupil engaged time (see Romberg, Small, & Carnahan, 1979, for a complete explication of

Table 41

Children in Each Cognitive Group in Each Class

Used in the Observation Study

	Sandy Bay Infant School		Waimea Heights Primary School			
	Class		Class			
	1	2	3	4	5	
Cognitive Group	Grade 1	Grade 2	Grade 3	Grade 3	Grade 3	Total
1	2	2				4
2	3	4		3		10
3	1	2	2	2	2	9
4			2	2	2	6
5,6			3	1	2	6
Totals	6	8	7	8	6	

the model). To test this model, data have been gathered on various components of the model in realistic classroom settings for several periods of time (see Romberg, Small, Carnahan, & Cookson, 1979, for a description of coding procedures used as well as detailed explanations of coding categories). From such data the relationship of the model to the reality of classroom instruction as it is observed in the field can be examined.

Summary of the Coding Procedure and Aggregation of Data

Data were collected on content covered and on certain teacher and pupil behaviors involved in the teaching and learning of mathematics using two procedures (complete details appear in Romberg, Collis, Buchanan, & Romberg, 1982). First, to estimate time spent on various mathematics objectives, teachers were asked to log the number of minutes on instruction in nine content areas spent for each target child. Seven of the nine areas dealt with aspects of learning to add and subtract. The "other arithmetic" area included time spent on both multiplication and division activities, and "other maths" encompassed all other activities such as measurement, fractions, or geometry. In Table 42, the percentage of total time spent on content area is presented. Overall, these data reflect the curricular emphasis common in these grades. Almost half of the time is spent on addition and subtraction. The emphasis obviously varies across grades. In grade 1, the highest percentage is on addition facts, numerousness and counting. In grade 2, basic facts for both addition and subtraction are still emphasized as are counting skills. And in grade 3, most of the emphasis is on computational algorithms. The only disappointing percentages are the little time spent on either writing sentences or

Table 42

Percentage of Time Spent on Mathematical Content Area by Grade--Teacher Log Data

Content Area	Grade 1	Grade 2	Grade 3
	(24 days, 50-60 min/day)	(25 days, 50-55 min/day)	(111 days, 30 min/day) (3 classes combined)
	Percent Time	Percent Time	Percent Time
Numerousness	14.3	6.4	4.5
Ordering	5.2	5.6	2.1
Basic Facts	15.5	13.3	4.0
(add)	(14.7)	(6.8)	(3.1)
(subtract)	(.8)	(6.5)	(.9)
Problem Solving	2.6	1.4	4.2
Sentence Writing	.8	.8	3.1
Algorithms	0	3.1	24.0
(add)	(0)	(3.1)	(13.4)
(subtract)	(0)	(0)	(10.6)
Counting	9.3	12.4	1.4
TOTAL addition and subtraction	47.7	50.2	44.3
Other Arithmetic	13.2	16.8	15.6
Other Maths	39.1	33.0	41.1

finding solutions to verbal problems. However, this differential emphasis is program-related, not child-related. For example, the reduction in percent of time spent on counting at grade 3 is not matched with the children's dismissal of its use to solve problems.

In fact, it can be argued that the structure of the program at grade 3 (emphasis on algorithms) presumes children have mastered most of the prerequisites (like counting and basic facts) and have acquired a high level of reasoning about numbers (be in cognitive Group 5,6). This is, of course, at odds with actual data on these students presented in the last two chapters.

Also, this description is fair in terms of the content included in the math curriculum in those schools, but it fails to capture important features of structure of those programs. In Sandy Bay Infant School, the program was filled with manipulative materials, lots of opportunity to explore independently or in small groups, use of learning stations, etc., and no basal text was used. However, in the third grades at Waimea Heights, a single text was followed and most activities involved paper and pencil seatwork.

Observed Data

Three trained observers gathered the data. These were the same persons who gathered data in studies 3 and 4. One observer worked at Sandy Bay Infant School and observed both the grade 1 and grade 2 classes. The other two worked at Waimea Heights Primary School where one observed two classes. Each was able to observe instruction in a class approximately 24 days during the observation period. At the schools, the observers sat in a class and over time became a fixture who did not detract either teacher or children.

Pupil action and teacher action data were gathered using an observation coding form. The exact nature of the data collected and the method used to gather it are described fully in the manual produced by the project staff to train observers (Romberg, Small, Carnahan, & Cookson, 1979).

In brief, student and teacher verbal behaviors were observed in each class on a sample of days. A time-sampling procedure was used in which each of six to eight "target" students was observed in a particular sequence at different moments throughout the observation period. The sequence in which the students were observed was fixed prior to the beginning of the observation period and was invariant while observations were taking place. The teacher was coded for instances of relevant verbal behavior each time a target student was observed. The observation of all six to eight students (along with the teacher six to eight times) represented a coding cycle. It was estimated that one minute was needed: (a) to observe the target student's behavior, (b) to observe the teacher, (c) to observe organizational aspects of the classroom, and (d) to code the appropriate categories on the observation form. The behavior to be coded consisted only of those activities the teacher and pupil were involved in precisely at the beginning of the one-minute time interval. Through this process, observer bias in sampling moments is minimized. The coding categories were used to record a description of what was occurring at that one instant for both the target student and the teacher. In this way, a series of "snap shots" would be obtained which would give a running account of what took place in the classroom for a particular observation period.

Observation for a class session began when mathematics instruction began, and ended when mathematics instruction for that class session ended. Not always did the beginning or ending of the observation period coincide with the beginning and ending of mathematics instruction as scheduled. As a result, two measures of time involved in mathematics class were obtained. Available time represented the scheduled time period in which mathematics instruction was to take place. Actual time, on the other hand, represented the amount of time mathematics instruction actually did occur. In most cases, the amount of time observing coincided closely with the measure of available time.

The basis data were aggregated in the form of frequency counts for each behavior category coded. For purposes of interpretation, the proportional occurrence of each behavior (based on total observed instances) is used. Data were aggregated separately for each class for the total period. The data give an overall picture of the teaching of mathematics in each class and yield estimates of how instructional factors affect engagement rates.

Data Aggregation and Analysis

The observational data gathered in this study have been summarized in terms of three categories: pupil actions, teacher behaviors, and teacher behavior-pupil engagement interactions. Pupil actions have been summarized in terms of engaged time; if engaged, whether it was on content or directions; grouping; interactions; and if interacting, with whom. Teacher behaviors have been summarized in terms of interactions, speaking to group, speaking on content or directions, questions, feedback and type of explanations.

Interactions of teacher behaviors and pupil engagement have been summarized in terms of whether pupils are engaged when the teacher is speaking, speaking to groups, listening, no teacher interactions, questioning, and provides information.

The plan for the analysis of the observational data was based on the fact there were two primary dimensions in the study: grade (or class) and cognitive group of the pupils. The raw data are observed minutes. Thus, the number of minutes and percent of time are aggregated in this analysis in five ways. First, we have aggregated data for all pupils with respect to grade. Second, since three different classrooms were being observed in grade 3, we have examined the data by class. Third, we examined the data for all students with respect to cognitive group. Fourth, we have examined the data by cognitive level within grade. And finally, we present the data in terms of cognitive level within class.

Pupil Actions

Grade. The data on pupil actions by grade is presented in Table 43. Significant engagement rate and grouping differences are apparent lecturian grades. Both are undoubtedly due to the differences in the structure of the curriculum in the schools. The high amount of time spent on small group and individual activities in grades 1 and 2 (85% and 68%, respectively) is consistent with the manipulative based, learning station approach at those grades. Similarly, 70% of the time spent in large group instruction at grade 3 is consistent with the text based direct instruction made in that school. Furthermore, it is interesting to note that the big difference in engagement is between grade 1 to grade 2 students who are following the same curriculum. In

Table 43

Observed Minutes and Percent of Time of Pupil Actions by Grade

Pupil Action	Grade 1		Grade 2		Grade 3	
	Minute	Percent	Minute	Percent	Minute	Percent
Engagement						
Engaged Time	559	55	771	71	1369	77
Off-task Time	449	45	317	29	403	23
Types of Engagement						
Content	488	89	656	86	1149	88
Directions	62	11	107	14	164	12
Grouping						
Individual	302	30	165	15	11	1
Small Group	553	55	583	53	524	29
Large Group	156	15	343	31	1259	70
Interactions						
Target Speaking	62	6	51	5	105	6
Target Listening	91	9	163	15	279	15
None	858	85	880	80	1427	79
Interaction Other Party						
Teacher	99	65	161	76	296	78
Pupil	48	31	36	17	77	20
Other Adult	6	4	16	8	6	2

fact, it was observed that in grade 1, many children spent a lot of time waiting for instructions about what to do next when they had completed an activity. By grade 2, this behavior was observed less frequently. Many students now proceeded to next task with little hesitation. Part of this change is probably due to increased student familiarity with behavioral expectations of the system and part is probably due to different teacher sensitivity to the situation.

Class. For grade 3, the data have been further subdivided into pupil actions by class as shown in Table 44. For comparative purposes, the data for grade 1 (class 1) and grade 2 (class 2) are shown again. Classes 3, 4, and 5 are all in grade 3. Class 4 is clearly different from the other two classes. Pupils in that class are off-task more of the time. Furthermore, if they are engaged, they are more likely to be engaged on directions, and if interacting are more likely to be interacting with other pupils. Differences in grouping are a function of curriculum since all third grade classes are similar on that dimension. Differences in engagement and interactions, however, are probably a function of the class or teacher.

Cognitive group. The number of minutes and percent of time coded to the five pupil action categories for all students in the cognitive groups are presented in Table 45. Overall, the percent of engaged time steadily increases across cognitive groups. Also, differences in grouping are striking with percent of time in large group instruction varying from 21% for Group 1 to 68% for Group 6 children. All other differences in percentage of time coded to the pupil action categories are not striking or of practical interest. However, these differences

Table 44

Observed Minutes and Percent of Time of Pupil Actions by Class

Pupil Action	Grade 1-Class 1		Grade 2-Class 2		Grade 3					
					Class 3		Class 4		Class 5	
	Minute	Percent	Minute	Percent	Minute	Percent	Minute	Percent	Minute	Percent
Engagement										
Engaged Time	559	55	771	71	402	98	650	64	317	90
Off-task Time	449	45	317	29	8	2	358	36	37	10
Types of Encouragement										
Content	488	89	656	86	364	95	496	79	289	97
Directions	62	11	107	14	21	5	135	21	8	3
Grouping										
Individual	302	30	165	15	6	1	0	0	5	1
Small Group	553	55	583	53	101	24	247	25	176	47
Large Group	156	15	343	31	317	75	750	75	192	51
Interactions										
Target Speaking	62	6	51	5	24	6	52	5	29	8
Target Listening	91	9	163	15	112	26	127	13	40	11
None	858	85	880	80	289	68	835	82	303	81
Interaction Other Party										
Teacher	99	65	161	76	122	92	119	67	55	81
Pupil	48	31	36	17	10	8	57	32	10	15
Other Adult	6	4	16	8	1	1	2	1	3	4

Table 45

Observed Minutes and Percent of Time of Pupil Actions by Cognitive Group

Pupil Actions	Cognitive Group 1		Cognitive Group 2		Cognitive Group 3		Cognitive Group 4		Cognitive Group 5	
	Minutes	Percent	Minutes	Percent	Minutes	Percent	Minutes	Percent	Minutes	Percent
Engagement										
Engaged Time	420	64	850	65	721	70	331	76	377	87
Off-task Time	237	36	460	35	310	30	106	24	56	13
Types of Engagement										
Content	361	86	690	83	634	90	282	88	326	91
Directions	57	14	140	17	68	10	37	12	31	9
Grouping										
Individual	167	25	201	15	104	10	0	0	6	1
Small Group	356	54	593	45	444	43	129	29	138	31
Large Group	135	21	510	39	496	48	317	71	300	68
Interactions										
Target Speaking	37	6	61	5	63	6	19	4	38	9
Target Listening	76	12	164	12	162	15	62	14	69	16
None	545	83	1090	83	825	79	367	82	338	76
Interaction Other Party										
Teacher	80	71	167	74	162	73	67	83	80	78
Pupil	24	21	46	20	55	25	14	17	22	21
Other Adult	9	8	12	5	6	3	0	0	1	1

in engagement and grouping are clearly confounded by the grade and class effects described earlier. This is due to the simple fact that at grade 1, 5 of 6 children observed were in cognitive Groups 1 and 2; at grade 2, 6 of 8 children were in Groups 1 and 2; and at grade 3, 12 of 21 children were in Groups 4, 5, and 6.

Cognitive level within class. To answer the basic question, do children with different cognitive capacities receive different instruction, the data for children within each class is presented. The data for children of different cognitive levels within class 1/grade 1 is presented in Table 46. Only the difference in time pupils interact with other pupils is significant between Group 1 and Group 3 children (24% to 45%).

The data for class 2/grade 2 children in different cognitive groups is presented in Table 47. As with grade 1, the only observable difference is in pupil interactions with other pupils (17% for Group 1 children and 32% for Group 3 children).

Tables 48, 49, and 50 contain the within class data for children in different cognitive groups for the three third-grade classes. The pictures of class 3 and class 5 show high engagement on content with virtually no differences between students. Class 4, on the other hand, exhibits much lower engagement with more time on directions for all students. Again, only pupil interactions with other pupils varies by cognitive level (31% for Group 2 children to 46% for Group 5,6 children).

Summary of the Pupil Action Data

Overall, this data suggests that differences in grouping of students are due to grade (structure of the curriculum). Grade 1 and

Table 46

Observed Minutes and Percent of Time of Pupil Actions

by Cognitive Group Within Class 1, Grade 1

Pupil Action	Cognitive Group 1		Cognitive Group 2		Cognitive Group 3	
	Minute	Percent	Minute	Percent	Minute	Percent
Engagement						
Engaged Time	260	60	189	51	110	54
Off-task Time	174	40	181	49	94	46
Types of Engagement						
Content	230	89	159	87	99	90
Directions	28	11	23	13	11	10
Grouping						
Individual	129	30	119	32	54	26
Small Group	235	54	197	53	121	59
Large Group	70	16	56	15	30	15
Interactions						
Target Speaking	26	6	24	6	13	6
Target Listening	41	9	30	8	20	10
None	368	85	318	85	172	84
Interaction Other Party						
Teacher	46	70	36	67	17	52
Pupil	16	24	17	31	15	45
Other Adult	4	6	1	2	1	3

Table 47

Observed Minutes and Percent of Time of Pupil Actions

by Cognitive Group Within Class 2, Grade 2

Pupil Action	Cognitive Group 1		Cognitive Group 2		Cognitive Group 3	
	Minute	Percent	Minute	Percent	Minute	Percent
Engagement						
Engaged Time	160	72	399	72	212	69
Off-task Time	63	28	158	28	96	31
Types of Engagement						
Content	131	82	336	85	189	90
Directions	29	18	57	15	21	10
Grouping						
Individual	38	17	82	15	45	14
Small Group	121	54	294	53	168	54
Large Group	65	29	179	32	99	32
Interactions						
Target Speaking	12	5	20	4	19	6
Target Listening	35	16	84	15	44	14
None	177	79	454	81	249	80
Interaction Other Party						
Teacher	34	72	87	84	40	65
Pupil	8	17	8	8	20	32
Other Adult	5	11	9	9	2	3

Table 48

Observed Minutes and Percent of Time of Pupil Actions by Cognitive Group Within Class 3, Group 3

Pupil Action	Cognitive Group 3		Cognitive Group 4		Cognitive Group 5,6	
	Minute	Percent	Minute	Percent	Minute	Percent
Engagement						
Engaged Time	144	98	80	96	178	99
Off-task Time	3	2	3	4	2	1
Types of Engagement						
Content	127	93	76	96	161	95
Directions	10	7	3	4	8	5
Grouping						
Individual	5	3	0	0	1	0
Small Group	33	22	20	23	48	26
Large Group	114	75	67	77	136	74
Interactions						
Target Speaking	8	5	2	2	14	8
Target Listening	47	31	16	18	49	29
None	98	67	69	79	122	66
Interaction Other Party						
Teacher	52	95	16	94	54	89
Pupil	3	5	1	6	6	10
Other Adult	0	0	0	0	1	1

Table 49

Observed Minutes and Percent of Time of Pupil Actions

by Cognitive Group Within Class 4, Grade 3

Pupil Action	Cognitive Group 2		Cognitive Group 3		Cognitive Group 4		Cognitive Group 5,6	
	Minute	Percent	Minute	Percent	Minute	Percent	Minute	Percent
Engagement								
Engaged Time	262	68	151	69	148	62	89	67
Off-task Time	121	32	101	40	92	38	44	33
Types of Engagement								
Content	195	76	119	83	112	77	70	80
Directions	60	24	24	17	33	23	18	20
Grouping								
Individual	0	0	0	0	0	0	0	0
Small Group	102	27	62	25	51	21	32	24
Large Group	275	73	187	75	187	79	101	76
Interactions								
Target Speaking	17	4	14	6	8	3	13	10
Target Listening	50	13	36	14	30	12	11	8
None	318	83	204	80	203	84	110	82
Interaction Other Party								
Teacher	44	66	33	67	29	76	13	54
Pupil	21	31	16	33	9	24	11	46
Other Adult	2	3	0	0	0	0	0	0

Table 50

Observed Minutes and Percent of Time of Pupil Action by Cognitive Group Within Class 5, Grade 3

Pupil Action	Cognitive Group 3		Cognitive Group 4		Cognitive Group 5,6	
	Minute	Percent	Minute	Percent	Minute	Percent
Engagement						
Engaged Time	104	87	103	90	110	92
Off-task Time	16	13	11	10	10	8
Types of Engagement						
Content	100	98	94	99	95	95
Directions	2	2	1	1	5	5
Grouping						
Individual	0	0	0	0	5	4
Small Group	60	48	58	48	58	46
Large Group	66	52	63	52	63	50
Interactions						
Target Speaking	9	7	9	7	11	9
Target Listening	15	12	16	13	9	7
None	102	81	95	79	106	84
Interaction Other Party						
Teacher	20	83	22	85	13	72
Pupil	1	4	4	15	5	28
Other Adult	3	13	0	0	0	0

grade 2 children are often working in small groups and individually for mathematics instruction while large group work is common in grade 3. Differences in engaged time are due to teachers or familiarity with the instructional pattern. And, only pupil interactions with other pupils are plausibly due to cognitive group of the children (with children in higher groups more likely to interact with others), but this behavior only occurs where such interactions are allowed and even then is infrequent.

Teacher Behaviors

The data for number of minutes and percent of time teacher actions were coded if first presented when the actions of target children by grade level were observed. Then, the teacher behaviors by class, by cognitive group, and by cognitive group/class interactions are presented.

Grade. The data on teacher behaviors by grade is presented in Table 51. The differential time spent by teachers explaining or giving directions vs. content is obviously a function of grade and is consistent with program expectations discussed earlier. Time spent on directions is inversely related to grade level.

Class. The data on teacher behaviors by class within grade 3 is shown in Table 52. The differences of speaking on content appear to be teacher or class specific. The differences between the first-grade teacher and two of the third-grade teachers on content remain significant. For example, for class 1 (grade 1), 51% of the time speaking is on content while for class 3 (grade 3), 82% is on content. But for class 4 (grade 3), again, 57% is on content. However, the percent of time teachers explain directions appears to be a grade

Table 51

Observed Minutes and Percent of Time of Teacher Behaviors by Grade

Teacher Behavior	Grade 1		Grade 2		Grade 3	
	Minute	Percent	Minute	Percent	Minute	Percent
Interaction						
Listening	187	17	206	18	216	11
Speaking	640	58	677	59	1238	64
None	276	25	254	22	485	25
Speaking/Large Group	91	14	209	31	313	25
Speaking/Small Group	82	13	65	10	227	18
Speaking/Individual	467	73	402	59	697	56
Speaking/Content	367	57	404	60	823	66
Speaking/Directions	268	42	256	38	347	28
Low Level Questions	135	12	157	14	338	17
Direction Related Questions	33	3	29	3	199	10
No Feedback	1006	91	1035	91	1819	94
Feedback/Individual	79	90	89	94	109	92
Low Information Feedback	97	100	101	98	115	93
High Information Feedback	0	0	2	2	9	7
Explaining Content	130	12	117	10	323	17
Explaining Directions	235	21	228	20	165	9

Table 52

Observed Minutes and Percent of Time of Teacher Behaviors by Class

					Grade 3					
Teacher Behavior	Grade 1-Class 1		Grade 2-Class 2		Class 3		Class 4		Class 5	
	Minute	Percent	Minute	Percent	Minute	Percent	Minute	Percent	Minute	Percent
Interaction										
Listening	187	17	206	18	55	12	129	12	32	8
Speaking	640	58	677	59	290	63	681	62	267	71
None	276	25	254	22	116	25	294	27	75	20
Speaking/Large Group	91	14	209	31	128	44	134	20	51	19
Speaking/Small Group	82	13	65	10	41	14	107	16	79	29
Speaking/Individual	467	73	402	59	121	42	439	64	137	51
Speaking/Content	367	57	404	60	239	82	391	57	193	71
Speaking/Directions	268	42	256	38	45	15	240	35	62	23
Low Level Questions	135	12	157	14	94	20	172	16	72	19
Direction Related Questions	33	3	29	3	22	5	125	11	52	14
No Feedback	1006	91	1035	91	434	94	1025	93	360	95
Feedback/Individual	79	90	89	94	24	92	71	93	14	83
Low Information Feedback	97	100	101	98	23	82	77	99	15	83
High Information Feedback	0	0	2	2	5	18	1	1	3	17
Explaining Content	130	12	117	10	96	21	139	13	88	23
Explaining Directions	235	21	228	20	26	6	126	11	13	3

effect since all three grade 3 teachers spend less time (6%, 11%, and 3%) than either the grade 1 or grade 2 teachers.

Cognitive group. The number of minutes and percent of time coded to six teacher behavior categories are presented in Table 53. Overall, three differences are striking across cognitive groups. First, the percent of time speaking to individual children decreases from 67% for Group 1 children to 53% for Group 5,6 children. Second, the percent of time teachers spend speaking about directions shifts from 39% for Group 1 children to 27% for Group 5,6 children. And in the same vein, when teachers are explaining the percent of time, explaining directions decreases from 22% for Group 1 children to 6% for Group 5,6 children. However, the later two differences are undoubtedly confounded by grade level.

Cognitive group within class. Tables summarizing the percent of time teacher behaviors were observed in each class in relationship to students in different cognitive groups are not presented here. For four of the classes (1, 2, 3, and 4), there were no striking differences in terms of time spent for different children. Only one significant difference was found. In class 5, the time spent by the teacher speaking on content decreased across groups from 82% to 66%.

In summary, while teacher behaviors vary considerably across teachers, differences are more due to grade, or individual teaching style, or grouping patterns within classes than they are to differential treatment of students with different levels of cognitive capacity. Teachers may treat some students differently than others, but this data suggests cognitive capacity is not the basis for such differentiation.

Table 53

Observed Minutes and Percent of Time of Teacher Behaviors by Cognitive Group

Teacher Behavior	Cognitive Group 1		Cognitive Group 2		Cognitive Group 3		Cognitive Group 4		Cognitive Group 5,6	
	Minute	Percent	Minute	Percent	Minute	Percent	Minute	Percent	Minute	Percent
Interaction										
Listening	127	19	231	16	139	13	50	10	62	14
Speaking	394	60	837	58	722	66	310	60	292	64
None	141	21	380	26	235	21	154	30	97	21
Speaking/Large Group	83	21	190	23	189	26	90	23	81	28
Speaking/Small Group	47	12	116	14	100	14	56	18	55	19
Speaking/Individual	264	67	529	63	433	60	184	59	156	53
Speaking/Content	233	59	479	57	475	66	204	66	203	69
Speaking/Directions	155	39	327	39	228	31	81	26	80	27
Low Level Questions	84	13	178	12	196	18	82	16	90	20
Direction Related Questions	12	2	78	5	68	6	51	10	52	11
No Feedback	592	83	1334	92	1018	93	481	93	427	94
Feedback/Individual	60	91	98	93	70	95	27	87	22	88
Low Information Feedback	69	99	114	100	75	94	31	91	24	92
High Information Feedback	1	1	0	0	5	6	3	9	2	8
Explaining Content	72	11	166	11	182	17	75	15	75	17
Explaining Directions	143	22	254	18	164	15	38	7	29	6

Teacher Behavior/Pupil Engagement Interactions

The number of minutes and percent of time teacher actions were coded and children were engaged is reported in this section. As with the previous sections, the data were first aggregated for children differing by grade, then class, cognitive group, and finally, cognitive group within class.

Grade. The data on pupil engagement for various teacher actions by grade is presented in Table 54. Overall pupil engagement when teachers are speaking increases from 59% in grade 1 to 78% in grade 3. Engagement when teachers are not speaking increases from 50% to 76%. Similarly, pupil engagement when there are no interactions increases from 42% to 78% across grades, as do all engagement rates related to teacher questioning and providing information.

Class. The information on pupil engagement when teachers performed certain actions is presented for all five classes in Table 55. As would be expected from previous analyses, class 4 in grade 3 is different from classes 3 and 5 in grade 3. Engagement rates in class 4 are lower in all categories than the other two classes. In fact, the grade level effect noted previously is in part an individual teacher effect, and certainly would be higher for grade 3 if class 4 were omitted.

Cognitive group. The overall data on time pupils in differing cognitive groups were engaged when teachers were doing different things is reported in Table 56. Many of the differences are striking. First, across groups, children increase in engagement when teachers are speaking from 65% of the time to 86%. Second, the overall pattern across groups is similar regardless of to whom the teacher is

Table 54

Observed Minutes and Percent of Time of Interactions of Teacher Behaviors and Pupil Engagement by Grade

Interaction	Grade 1		Grade 2		Grade 3	
	Minute	Percent	Minute	Percent	Minute	Percent
Teacher Speaking/						
Pupil Engaged	356	59	463	70	919	78
Pupil Off-task	245	41	197	30	259	22
Pupil Engaged When Teacher Speaking to:						
Individual	253	57	265	68	502	74
Small Group	51	67	45	69	160	80
Large Group	52	63	153	75	256	86
Not Speaking	203	50	308	72	449	76
Pupil Engaged When Teacher:						
Listening	104	61	151	76	152	72
Pupil Engaged When:						
No Interactions	99	42	157	69	295	78
Pupil Engaged When Teacher Asks:						
Low Level Questions	75	60	108	71	263	81
High Level Questions	8	67	19	83	42	95
Questions About Directions	15	52	20	69	133	70
Pupil Engaged When Teacher Provides:						
Low Information Feedback	44	48	67	68	90	80
Positive Feedback	32	54	56	72	52	87
Information About Content	83	68	89	77	255	82
Explains Directions	131	58	149	67	114	72

Table 55

Observed Minutes and Percent of Time of Interactions of Teacher Behaviors and Pupil Engagement by Class

Interaction	Grade 1-Class 1		Grade 2-Class 2		Grade 3					
					Class 3		Class 4		Class 5	
	Minute	Percent	Minute	Percent	Minute	Percent	Minute	Percent	Minute	Percent
Teacher Speaking/										
Pupil Engaged	356	59	463	70	263	99	437	66	219	88
Pupil Off-task	245	41	197	30	2	1	226	34	31	12
Pupil Engaged When Teacher Speaking to:										
Individual	253	57	265	68	109	99	275	63	118	87
Small Group	51	67	45	68	31	100	69	59	60	90
Large Group	52	63	153	75	123	99	92	74	41	85
Not Speaking	203	50	308	72	138	96	213	62	98	94
Pupil Engaged When Teacher:										
Listening	104	61	51	76	51	98	73	57	28	90
Pupil Engaged When:										
No Interactions	99	42	157	69	88	95	141	65	66	96
Pupil Engaged When Teacher Asks:										
Low Level Questions	75	60	108	71	86	99	116	76	61	92
High Level Questions	8	67	19	83	23	100	1	100	18	90
Questions About Directions	15	52	20	69	21	100	70	60	42	82
Pupil Engaged When Teacher Provides:										
Low Information Feedback	44	48	67	68	22	100	56	73	12	92
Positive Feedback	32	54	56	72	20	100	23	82	9	75
Information About Content	83	68	89	77	86	100	95	69	74	86
Explains Directions	131	58	149	67	24	96	80	66	10	91

Table 56

Observed Minutes and Percent of Time of Interactions of Teacher Behaviors and Pupil Engagement by Cognitive Group

Interaction	Cognitive Group 1		Cognitive Group 2		Cognitive Group 3		Cognitive Group 4		Cognitive Group 5,6	
	Minute	Percent	Minute	Percent	Minute	Percent	Minute	Percent	Minute	Percent
TeacherSpeaking/										
Pupil Engaged	254	64	518	66	509	73	216	76	241	86
Pupil Off-task	137	35	267	34	188	27	69	24	40	14
Pupil Engaged When Teacher Speaking to:										
Individual	174	67	306	61	289	69	128	73	123	79
Small Group	26	55	78	71	75	82	33	72	44	92
Large Group	54	65	133	77	145	78	55	87	74	95
Not Speaking	166	62	332	63	211	63	115	76	136	89
Pupil Engaged When Teacher:										
Listening	94	75	145	68	81	60	35	74	52	87
Pupil Engaged When:										
No Interactions	72	51	188	60	130	66	79	76	82	91
Pupil Engaged When Teacher Asks:										
Low Level Questions	51	62	114	68	141	75	60	77	80	92
High Level Questions	7	88	15	75	17	85	15	100	15	94
Questions About Directions	8	67	41	59	46	70	34	68	39	78
Pupil Engaged When Teacher Provides:										
Low Information Feedback	43	62	68	62	50	70	20	71	20	83
Positive Feedback	32	65	46	66	37	77	10	83	15	83
Information About Content	53	74	106	68	147	82	57	83	64	90
Explains Directions	87	61	159	67	101	63	25	69	22	79

speaking, and even when the teacher is not speaking (62% engagement to 89%). Third, in the same manner, pupil engagement increases from 51% for Group 1 children to 91% for Group 5,6 children, when there were no teacher interactions. Finally, the same pattern of increase in engagement is apparent when teachers question students or provide information. However, as in the previous analyses, these are the same differences found across grade levels.

Cognitive level within class. The engagement data for children of differing cognitive levels within each class was also calculated. Although there is some variation in engagement in each class for children of differing cognitive levels, no discernable pattern of differences in any class was apparent. Thus, tables summarizing this data are not presented.

In summary, the data relating pupil engagement to type of teacher behavior suggest that differences are due to grade level and teacher style and not to differences in cognitive capacity among the students within each class.

Conclusions

The question raised at the beginning of this chapter: Do children who differ in cognitive capacity receive different instruction?, now can be answered. No! At least that is the case for the sample of students in the five classes observed in this study.

Nevertheless, the data from this study provide several interesting insights about mathematics instruction. First, teachers tend to organize and teach mathematics based on school traditions. Differences in content emphasis and patterns of grouping students are based on program expectations within schools. In particular, the differences in pupil actions and teacher action from grades 1 and 2 to grade 3 reflect a shift in emphasis and organization of activities. Sandy Bay Infant School (grades 1 and 2) has an open, activity oriented program. Waimea Heights, on the other hand, is a "primary" school where instruction is more formal and direct. Hence, the overwhelming grade level effect on pupil actions, teacher actions, and pupil engagement is to be expected.

Second, the mathematics program within schools is not related either to how students work problems or their capacity to reason. Third, there are important differences between teachers who are trying to do the same thing. Classes 3 and 5 in grade 3 clearly reflect good teaching following direct instruction approach. Children are on task in large or small groups. Class 4, on the other hand, while following the same program, is not a successful class.

Fourth, the only interesting pupil behavior related to cognitive capacity is the tendency for children in higher groups to interact with other pupils more often when there is an opportunity to interact.

Chapter 6

SUMMARY, CONCLUSIONS, AND IMPLICATIONS

The question being investigated in this set of five studies was Do children who differ in cognitive capacity learn to add and subtract differently? In asking this question, we assumed that children's performance on addition and subtraction problems was related both to their cognitive capacity and to classroom instruction. This series of studies were reported from four different intellectual perspectives so that each study would shed light on a different aspect of the question. Then, by putting the information from each together, we hoped to answer the basic question.

In retrospect, we believe the picture which has evolved from these studies about how children learn to add and subtract is both interesting and provocative, but not at all clear. This chapter summarizes what we learned and specifies the strengths and weaknesses of each of the studies. Finally, implications are suggested to other researchers, to curriculum developers, and to teachers. We have chosen to organize this discussion under five headings: cognitive capacity, solving verbal addition and subtraction problems, using the concepts and skills of addition and subtraction, the influence of instruction on addition and subtraction performance, and final reflections.

Cognitive Capacity

The original question assumes that young children differ in their cognitive capacity to deal with mathematical information and that available psychometric techniques would yield clusters of students into groups so that members in each group had similar scores from tests related to mathematics. First, a set of tests measuring short-term memory capacity (M-space) were administered. Second, a set of developmental psychological tests were given to the same children. Data from these tests were used to empirically derive six groups of students. The groups were based on the M-space tests. The developmental tests were then used to assist in describing the differences between the groups.

Cognitive Group 1 children have limited memory capacity (M-space level 1), are incapable of handling most quantitative tests, can serially count but have no sophisticated counting strategies, and can only deal with qualitative comparisons and transformations at a moderate level.

Cognitive Group 2 children have larger memory capacities (M-space level 2), have no difficulty with qualitative comparisons (They can preserve correspondence after rearrangement of sets and overcome perceptual distractions.), and can determine whether sets are larger or smaller if an object has been put with or taken from particular sets. However, the quantitative skills of these children are limited. They can count sets, but have no sophisticated counting strategies, and are unable to handle transitivity and rearrangement problems.

These first two groups are distinct from each other and distinct from the remaining four groups. The final four groups, both psychometrically and logically, are more similar to each other than they are different from each other in that all members have a memory capacity of level 3 or 4 and have sophisticated counting strategies.

Cognitive Group 3 children differ from Cognitive Group 4 children only on measures of transitivity and transitivity under rearrangement. Group 3 and 4 children differ only from Group 5 and Group 6 children on the class inclusion test. The difference between Group 5 children and Group 6 children is only on one measure of memory capacity; in all analysis we combined both Groups 5 and 6.

The data gathered and analyzed with respect to cognitive capacity suggested the following six propositions. First, a global qualitative versus quantitative distinction is apparent in children's mathematical thinking in the early school years. Second, M-space level seems to be related to the developmental sequences in the preschool to early elementary years in mathematically-related tests. Third, the development of reasoning appears to be: comparison -- qualitative -- correspondence -- quantitative -- logical operations. Fourth, an M-space level of 1 is enough for handling simple comparison tasks. Fifth, an M-space level of 2 is enough for qualitative correspondence and is a prerequisite for the development of most number skills. And sixth, an M-space level of 3 seems to be necessary for success in sophisticated counting tasks and probably is necessary for the development of addition and subtraction.

Problems and recommendations. The data indicate there are children who differ significantly in their ability to handle early mathematical tasks. However, the approach that we took is purely empirical. It is not based on any theory of how mathematical information is actually processed. The next step would be to use a theoretic model of cognitive processing such as that proposed by Campione and Brown (1978) which distinguishes between the "architectural" features of cognition (memory capacity, automaticity, speed of processing, etc.) and "executive" aspects of cognitive processing (metacognition, hueristics, schema in long-term memory, etc.). Using such a model would enhance our understanding of cognitive capacity in a more powerful fashion than the psychometric approach followed in the study could ever do. Nevertheless, we uncovered clear evidence that there are groups of children who differ significantly in the way in which they process information. Furthermore, three sets of measures used were important. First, we found memory capacity to be most important in identifying groups with differing cognitive capacities. Unfortunately, the instruments used to assess this underlying trait leave much to be desired. In particular, on the Mr. Cucui test, children can organize information by "chunking" it (e.g., left side of the body, head, and so on). As a result, higher M-space levels are indicated because a smaller part of memory is being used for more information. This phenomena is well known in the literature, but to separate "chunking" from actual M-space is difficult. We believe that the four tests indicate M-space level 1, 2, and 3 relatively accurately. However, memory capacity levels above 3 in many cases may be due to "chunking" of information. Nevertheless, we are convinced that memory capacity is an important feature and would strongly suggest other researchers measure memory capacity of their subjects.

The second set of tests which distinguished groups were the "counting forward" and "counting back" tests. Sophisticated counting skills are important, as demonstrated in Chapter 3. When students solve verbal addition and subtraction problems, many use such skills. We recommend that such tests be used in other research. Also, other tests which measure different counting skills (simple counting, counting on, counting back, counting all, etc.) should also be developed and used.

Finally, the "class inclusion" test distinguished the groups of students. The relationship of "class inclusion" to how children work certain problems (particularly the part-part-whole problems) is not at all clear. We recommend the development and study of other tests which assess the way in which individuals logically reason about phenomena.

Solving Verbal Addition and Subtraction Problems

One indication that students have learned to add and subtract is that they can solve simple verbal problems. For such problems, it is expected that children can write an addition or subtraction sentence about the problem and use learned addition or subtraction concepts or skills to find the appropriate answer. In Chapter 3, we examined both the performance of students (the number of questions they were able to answer correctly) and the strategies they used to solve a variety of addition and subtraction problems. The data were gathered in interviews of each child on several occasions in which six problems were given to each student at two or three of four levels of difficulty, determined by the size of numbers in the problem. The results described in Chapter 3 indicate that there was considerable variability in the children's ability to solve the variety of verbal problems and the strategies they used to solve those problems. The overall performance of the different groups of students on the tasks was relatively high. Of the B level problems, 72% were answered correctly; of the C level problems, 72% were answered correctly; and of the D and E level problems, 67% were answered correctly. However, there was considerable variability in both performance and strategies which was influenced by several factors: the semantics of the problem, the size of the numbers in the problem, the implied operation in the problem, the grade level of the child, and the cognitive capacity of the child.

Table 57 summarizes the level of performance across all items for the six different semantic types. In general, the results support the conclusion of Greeno and Riley (1981) that change problems are in general easier than combine problems which in turn are easier than compare problems. However, implied operation as well as semantics clearly makes a difference.

The most striking findings of the study on both performance and strategies were for children in different cognitive groups. The performance and strategies used by the children in each cognitive group are summarized on the following pages. The percent correct is noted only if at least two-thirds of the tasks within a semantic category were answered correctly. Similarly, to highlight the strategies used by students in each particular group, percentages are indicated only if the strategy was used at least 20% on the same semantic set of problems.

Table 57

Frequency and Percent Correct for Each Task on B, C, D, E Level Items and Over All Levels

Task	Level B		Level C		Level D,E		All Levels	
	Frequency	Percent	Frequency	Percent	Frequency	Percent	Frequency	Percent
1 Change/Join (+)	85	85	201	78	125	82	411	81
2 Change/Separate (-)	79	79	184	71	89	58	352	69
3 Combine/Part Unknown (-)	68	68	190	74	88	58	346	68
4 Combine/Whole Unknown (+)	80	80	194	75	115	76	389	76
5 Compare (-)	41	41	157	58	90	59	288	56
6 Change/Join, Change set Unknown (-)	77	77	191	74	105	68	373	73

The summary information on the performance and use of strategies for the children in cognitive Group 1 is presented in Table 58. This group of children performed satisfactorily only on 3 of the 12 tasks, only on the B set of tasks, and only on the 3 tasks which can easily be solved by direct modeling. The strategies these students used, with one exception, were either inappropriate or direct modeling. The one exception was on task 6 at the B level when "counting on" was used 37% of the time.

Overall, this behavior clearly reflects the cognitive capacity of these children. They had low memory capacity, lack of systematic counting skills, and were only able to directly model the problems. Also, the compare task, which requires more memory capacity, was impossible for the children; inappropriate strategies were used on the B and C level compare tasks 83% and 93% of the time, respectively.

The summary information on the performance and use of strategies for students in cognitive Group 2 is shown in Table 59. This group of children could find answers satisfactorily on both the B and C sets of problems, with the exception of "compare" tasks. Although the level of performance was slightly lower on the C set than the B set, the pattern of the performance was very similar. However, on the D and E sets (larger numbers), only task 1 was answered at a level of performance which is satisfactory.

The strategy information is consistent with the level of cognitive capacity demonstrated for this group of students. Direct modeling was the most frequently used strategy for both the B and C level problems, although routine mental operations are becoming commonplace with the small number of problems in the B set. The

Table 58

Performance and Common Use of Strategies for Cognitive Group 1

Task	Percent Correct	Direct Modeling	Counting Sequences	Routine Mental Operation	Inappropriate
B Level					
1 Change/Join (+)	77	53			
2 Change/Separate (-)	70	57			33
3 Combine/Part Unknown (-)		40			50
4 Combine/Whole Unknown (+)	70	57			20
5 Compare (-)					83
6 Change/Join, Change set Unknown (-)			37		37
C Level					
1 Change/Join (+)		37			63
2 Change/Separate (-)		40			60
3 Combine/Part Unknown (-)		37			60
4 Combine/Whole Unknown (+)		37			63
5 Compare (-)					93
6 Change/Join, Change set Unknown (-)					80

Table 59

Performance and Common Use of Strategies for Cognitive Group 2

Task	Percent Correct	Direct Modeling	Counting Sequences	Routine Mental Operation	Inappropriate	Algorithm
B Level						
1 Change/Join (+)	85	44		33		
2 Change/Separate (-)	81	52		31		
3 Combine/Part Unknown (-)	77	50		27		
4 Combine/Whole Unknown (+)	83	48		27		
5 Compare (-)					58	
6 Change/Join, Change set Unknown (-)	85		36	38		
C Level						
1 Change/Join (+)	72	42				
2 Change/Separate (-)	68	43				
3 Combine/Part Unknown (-)	71	42				
4 Combine/Whole Unknown (+)	68	45	24			
5 Compare (-)					53	
6 Change/Join, Change set Unknown (-)	70	28	26		26	
D,E Level						
1 Change/Join (+)	67				25	50
2 Change/Separate (-)					33	33
3 Combine/Part Unknown (-)					41	
4 Combine/Whole Unknown (+)					42	33
5 Compare (-)					50	
6 Change/Join, Change set Unknown (-)		21			29	

compare problems are still in most cases impossible. Inappropriate strategies were coded in over half of the trials over all problems. The only systematic counting strategies are used on task 6 on both B and C levels and task 4 on the C level problems. Finally, for the problems with larger numbers (D and E), inappropriate strategies are most frequent on all tasks except task 1. Only on this task do these children choose to use an algorithm. However, these students very often made errors in the use of algorithm when solving these problems.

The summary information for the children in cognitive Group 3 appears in Table 60. Their overall level of performance is quite satisfactory on all tasks at the B and C levels; there is still some difficulty associated with task 5. For the D and E set, only on tasks 1, 4, and 6 is performance satisfactory. Direct modeling is a reasonable strategy, particularly on the small B level problems. Counting strategies, however, and routine mental operations are also being used with small number problems. Sophisticated counting strategies were used on all C level tasks and on three of the D and E level tasks. Also, a fairly high frequency of inappropriate strategies were apparent at the D and E level tasks. The Group 3 students chose to use algorithms for the D and E problems less frequently than did the Group 2 children.

The summary information for cognitive Group 4 children appears in Table 61. Not surprisingly, the performance and choice of strategies of these children differs very little from the Group 3 students. Counting strategies and routine mental operations are used on the C level problems. Direct modeling and counting strategies are being

Table 60

Performance and Common Use of Strategies for Cognitive Group 3

Task	Percent Correct	Direct Modeling	Counting Sequences	Routine Mental Operation	Inappropriate	Algorithm
B Level						
1 Change/Join (+)	100	33		39		
2 Change/Separate (-)	89	56		33		
3 Combine/Part Unknown (-)	89	50		28		
4 Combine/Whole Unknown (+)	89	56		33		
5 Compare (-)	67			44	33	
6 Change/Join, Change set Unknown (-)	94		44	39		
C Level						
1 Change/Join (+)	72		35	29		
2 Change/Separate (-)	68	23	26			
3 Combine/Part Unknown (-)	71	27	26	26		
4 Combine/Whole Unknown (+)	80		42	26		
5 Compare (-)	74		36	24	23	
6 Change/Join, Change set Unknown (-)	82		32	35		
D,E Level						
1 Change/Join (+)	89			34		27
2 Change/Separate (-)		34				29
3 Combine/Part Unknown (-)		29	23		25	
4 Combine/Whole Unknown (+)	70			27	23	29
5 Compare (-)			36		25	
6 Change/Join, Change set Unknown (-)	66		36		25	

Table 61

Performance and Common Use of Strategies for Cognitive Group 4

Task	Percent Correct	Direct Modeling	Counting Sequences	Routine Mental Operation	Inappropriate	Algorithm
C Level						
1 Change/Join (+)	91		34	41		
2 Change/Separate (-)	77		39	27		
3 Combine/Part Unknown (-)	75		32	25		
4 Combine/Whole Unknown (+)	86		25	39		
5 Compare (-)	86	20	27	32		
6 Change/Join, Change set Unknown (-)	84		23	45		
D,E Level						
1 Change/Join (+)	78			38		38
2 Change/Separate (-)		31				26
3 Combine/Part Unknown (-)		31			31	
4 Combine/Whole Unknown (+)	74					45
5 Compare (-)		21	31		24	
6 Change/Join, Change set Unknown (-)	67	29	31		21	

used with the D and E problems. There is some increase in the use of algorithms on the two addition problems.

The summary information of children in cognitive Groups 5 and 6 is shown in Table 62. These children solve all problems at a satisfactory level. Counting strategies and routine mental operations are used to find most solutions. However, on the D and E simple subtraction problems, tasks 2 and 3, direct modeling is frequently used. Routine mental operations and algorithms are only used with the three easiest tasks.

What is more important in this data is that a child's decision to use a particular strategy is dependent on several factors including the semantics of the problem, the size of the numbers, and the implied operation. Furthermore, the availability or use of a strategy appears to be dependent upon memory capacity.

Overall, five general observations from the data relate to our understanding how children learn to solve such problems. First, the frequent and persistent use of inappropriate strategies implies either an unwillingness of some students to engage in the task, or a lack of the memory capacity to use a particular strategy. We agree with DeCorte and Verschaffel (1981) that some students fail to understand they are to find an answer to a particular problem. However, we believe most students try but lose track of information. For example, Group 1 and Group 2 children do not have systematic counting strategies available to them to solve many of the problems. Thus, when they try, they may get mixed up and are unable to complete the task. We recommend a more careful investigation of the use of inappropriate strategies across tasks. By examining the use of

Table 62

Performance and Common Use of Strategies for Cognitive Group 5,6

Task	Percent Correct	Direct Modeling	Counting Sequences	Routine Mental Operation	Inappropriate	Algorithm
			C Level			
1 Change/Join (+)	95		33	50		
2 Change/Separate (-)	95		48	24		
3 Combine/Part Unknown (-)	98		31	48		
4 Combine/Whole Unknown (+)	95		38	48		
5 Compare (-)	93		48	36		
6 Change/Join, Change set Unknown (-)	98		43	45		
			D,E Level			
1 Change/Join (+)	88			32		39
2 Change/Separate (-)	74	31				38
3 Combine/Part Unknown (-)	83	33	40			
4 Combine/Whole Unknown (+)	93			26		50
5 Compare (-)	71		60			
6 Change/Join, Change set Unknown (-)	90		62			

inappropriate strategies in future studies, a better sense of the difficulties some children have might be found.

Second, direct modeling (the use of chips or fingers to represent sets) is the earliest and easiest strategy used by students. It is particularly appropriate for B level tasks 1, 2, and 4 where the change or combination can be physically represented. Also, the action preserves all the important data; prior data need not be remembered. The strategy is appropriate for task 3 and task 6, but it requires additional memory storage to remember the whole the the original part. Finally, direct modeling could be used with comparison problems, but it requires even more memory storage. Even with large number problems where physical modeling becomes more cumbersome, it still is an appropriate strategy. Many students appear to follow a "when in doubt one can always model" strategy for solving many problems. Even children in Group 5,6 at third grade physically modeled many of the large number problems to find answers. This suggests the importance of being familiar with efficient procedures; although children in Group 5,6 have exhibited sophisticated counting strategies, know basic facts, and can perform addition and subtraction algorithms with efficiency, they still directly model some problems.

Third, direct modeling, for many children, is replaced either by the use of systematic counting strategies or by the use of routine mental operations. Counting strategies may be learned before routine mental operations; the choice of strategy is dependent upon the size of numbers involved in the problem. At all levels, for all cognitive groups, the B level problems were solved using routine mental operations rather than counting strategies. Only for task 6 at the B level was counting the dominant strategy. The C, D, and E problems were more likely to be done using sophisticated counting strategies. Furthermore, only on task 1 (combine/join) were routine mental operations used with large number problems.

Fourth, the use of addition and subtraction algorithms for many children is perceived as a cumbersome procedure for finding answers. Only Group 2 children, who are limited in their knowledge of counting strategies or routine mental operations, use algorithms frequently, and they make many errors. Students at higher cognitive levels may see that algorithms are appropriate but know of and are comfortable in using other strategies. The children's teachers had expected students to write the mathematical expression and use the algorithms to solve problems on the D and E tasks. Most instruction had been on addition and subtraction algorithms and the children's performance was reasonably good.

Fifth, it is apparent that the way in which students solve the problems is not directly related to classroom instruction. In grade 2, most instruction was on addition and subtraction facts (use of routine mental operations), but direct modeling and counting skills were used by most students to solve the problems. In grade 3, most of the time was spent on algorithms, but they were not used.

Problems and recommendations. First, the sample of items, six tasks at each of the four levels, does not encompass the variety of addition and subtraction problems. Nesher, Greeno, and Riley (1982) list 14 types. Use of a more comprehensive set of problems would give us a better picture of the overall development of strategies across tasks. Second, the small number of students and the method of selection for this study is limiting. Studies with a larger number of subjects are in order. Third, although some longitudinal data were gathered, there was relatively little change in performance over the three-month time period. Studies of longer duration should be carried out. The cross-sectional data indicate possible changes, but a better longitudinal picture is desirable.

Finally, these data need to be re-examined in light of the recent proposed theory of the development of semantic categories for addition and subtraction (Nesher, Greeno, & Riley, 1982). Our data suggest that the decision sequence children use to select a strategy is more complex than suggested by that theory.

Using the Concepts and Skills of Addition and Subtraction

Since most textbooks for the teaching of arithmetic skills do not emphasize the solution of verbal problems, we also examined the growth and level of performance on the concepts and skills of addition and subtraction. This study is reported in Chapter 4. A set of achievement monitoring tests which measured a variety of mathematics objectives was administered at each grade level. Instruction at each grade level had an effect on some objectives over the autumn term. In grade 1, at the start of the school year, students were unable to solve most problems; by the end of the term, addition skills had improved dramatically, although the same could not be said for subtraction. In grade 2, although instruction had an effect, increases in performance were not as apparent for many objectives. In grade 3 there was clear improvement on many of the objectives. In particular, the level of performance on the addition and subtraction algorithms improved dramatically. Thus, growth within each grade on some aspects of addition and subtraction is very clear. However, improvement was not uniform across different concepts and skills, and the overall level of performance on many objectives was not high. Instruction does not seem to be related very systematically to the level of performance. Thus, in spite of the fact that overall performance on place value, knowledge of addition and subtraction facts, and writing number sentences was not high, time was not allocated for instruction on those topics. For example, in third grade, most students still were having trouble with writing open sentences and knowledge of basic addition and subtraction facts. Yet almost no time was allocated for instruction in these areas.

Performance by students in cognitive groups within grades differ. Group 1 children in grade 1 struggled with many of the objectives, while the Group 2 students increased in performance over all of the objectives. The children in the higher cognitive groups performed better than children in lower cognitive groups. Overall, children who differ in cognitive processing capacity performed differently regardless of specific objectives, instructional emphasis, or grade.

Thus, while teaching and pupil experience accounted for some of the differences between children, the level of performance appears to be consistent with cognitive processing capacity.

Influence of Instruction on Addition and Subtraction

The final study of the series of studies reported in this monograph looked at the question Do children with different cognitive capacities receive different instruction?

Observational data was collected on allocated time, pupil engagement, and teacher actions in relationship to pupil behavior.

The findings of this study are not dramatic. However, what is portrayed is perhaps too typical of how instruction is carried out in many schools. First, on content covered, about 50% of the total mathematics time in each grade is spent on addition and subtraction. In grade 1, primary emphasis is on addition facts, numerousness and counting. In grade 2, basic facts for both addition and subtraction are taught. And in grade 3, computational algorithms are stressed. The amount of time spent on various mathematics topics is not related to the level of performance of students on those topics. What pupils do in classrooms is dictated by the grade level and the structure of the curriculum. In grades 1 and 2, children were working in small groups and individually for mathematics instruction while large group work was common in grade 3. Differences in pupil engaged time are due to teachers or student familiarity with the instructional pattern. Only the number of pupil interactions with other pupils is possibly due to the cognitive group to which the children belong. Teacher behaviors reflect grade level and individual teaching style. Certainly, cognitive capacity is not the basis of differentiation between students in these classrooms.

The data in the last two studies clearly indicate that children improve due to instruction on basic facts and algorithmic performance. What teachers do in classrooms varies, but within classrooms, they teach basically the same way to all children. What children learn appears to be consistent with their level of cognitive processing and with the content covered in each grade. The emphasis within classrooms seems to be on some routine procedures (basic facts and algorithms) but not on others such as sentence writing, counting, or direct modeling of problems. The emphasis is on finding answers regardless of the procedure. Nothing is done to relate the semantics of various verbal problems to instruction in arithmetic.

Finally, there is no evidence that instruction attempts to build on or change the strategies that students are using to solve such problems. In fact, instruction seems to proceed without consideration of the level of performance of individual children in the classrooms.

Final Reflections

In concluding this monograph, seven thoughts come to mind. First, the information processing approach to the study of how children solve a variety of addition and subtraction problems appears to provide a basis for a better understanding of the process of acquiring concepts and skills acquired and using them to solve problems. Our clustering of children into cognitive groups should be viewed as a rough first approximation to a more elegant description of capacity. Second, for students struggling with the basic ideas (students in our Groups 1 and 2), a more careful analysis of inappropriate strategies needs to be done. Third, the most interesting set of data is on the strategies that children use, not on performance. Longitudinal data on change of strategies by specific children should be gathered. Fourth, curricula to be more effective must be organized and sequenced differently. Although the ideal organization and sequence for teaching addition and subtraction skills is not yet clear, more work on writing sentences and counting strategies is called for. Also, perhaps specific routines such as addition and subtraction facts or algorithmic procedures need to be initially taught without trying to tie them to problems until they have been mastered. Students could build the bridge from verbal problems to use of algorithms later.

Fifth, students need much more opportunity to work with verbal problems and to represent such problems with mathematical expressions. This procedure of modeling a problem situation with mathematical sentence is a very important skill throughout all mathematics. Sixth, while we believe learning routine procedures is important, they only became important in the eyes of children when they are seen as efficient and students feel confident in their use in solving problems. Seventh, children differ in their capacity to handle a variety of mathematical problems. Instruction should begin where children are. Teachers should take into account the strategies and procedures children use to solve problems and build upon those capacities.

In conclusion, our intent was to incorporate data from different perspectives to study how children learn to add and subtract. The picture which emerges is of children struggling to learn a variety of important concepts and skills. Some children are limited by their capacity to handle information. Most are able to solve a variety of problems by using invented strategies which have not been taught. They dismiss or fail to see the value of the taught procedures to solving those problems. The capacity of children for processing information, the procedures students have invented to solve a variety of problems, and the way in which instruction in schools is carried out are not in tune with each other. The challenge in the future is to change this fact. Our goal is to orchestrate instruction so that it is in harmony with children's capacities and their strategies.

References

Bachelder, B. L., & Denny, M. R. A theory of intelligence: I. Span and the complexity of stimulus control. *Intelligence*, 1977, *1*, 237-256.

Berliner, D. *Impediments to the study of teacher effectiveness*. Paper presented at the meeting of the National Association for Research in Science Teaching, Los Angeles, Calif., March 1975.

Biggs, J. B., & Collis, K. F. *Evaluating the quality of learning: The SOLO taxonomy*. New York: Academic Press, 1982.

Bruner, J. S. *Toward a theory of instruction*. New York: Norton, 1966.

Buchanan, A. E., & Romberg, T. A. *Instrumentation for the achievement monitoring component of coordinated study #1*. Madison: Wisconsin Center for Education Research, 1983.

Campbell, D. T., & Stanley, J. C. Experimental and quasi-experimental designs for research on teaching. In N. L. Gage (Ed.), *Handbook of research on teaching*. Chicago: Rand McNally, 1963.

Campione, J. C., & Brown, A. L. Twoard a theory of intelligence: Contributions from research with retarded children. *Intelligence*, 1978, *2*, 279-304.

Carpenter, T. P., & Moser, J. M. *An investigation of the learning of addition and subtraction* (Theoretical Paper No. 79). Madison: Wisconsin Research and Development Center for Individualized Schooling, 1979.

Carpenter, T. P., & Moser, J. M. The development of addition and subtraction problem-solving skills. In T. P. Carpenter, J. M. Moser, & T. A. Romberg (Eds.), Addition and subtraction: A cognitive perspective. Hillsdale, N.J.: Erlbaum, 1982.

Carpenter, T. P., Moser, J. M., & Romberg, T. A. (Eds.). Addition and subtraction: A cognitive perspective. Hillsdale, N.J.: Erlbaum, 1982.

Case, R. Learning and development: A neo-Piagetian interpretation. Human Development, 1972, 15, 339-358.

Case, R. Gearing the demands of instruction to the developmental capacities of the learner. Review of Educational Research, 1975, 45, 59-97.

Case, R. A developmentally based theory and technology for the design of effective instruction. Review of Educational Research, 1978, 48, 439-463. (a)

Case, R. Implications of developmental psychology for the design of effective instruction. In A. M. Lesgold, J. W. Pellegrino, S. D. Fokkema, & R. Glasser (Eds.), Cognitive psychology and instruction. New York: Plenum, 1978. (b)

Case, R., & Globerson, T. Field independence and central computing space. Child Development, 1974, 45, 772-778.

Case, R., & Kurland, D. Development of the counting span test, unpublished manuscript, Ontario, 1978.

Case, R., Kurland, M., Daneman, M., & Emmanuel, P. Operational efficiency and the growth of M-space. Paper presented at the biennial meeting of the Society for Research in Child Development, San Francisco, 1979.

Collis, K. F. A study of concrete and formal reasoning in school mathematics. Australian Journal of Psychology, 1971, 25, 121-130.

Collis, K. F. Cognitive development and mathematics learning. London: Chelsea College, University of London, 1974. (a)

Collis, K. F. The development of a preference for logical consistency in school mathematics. Child Development, 1974, 45, 978-983. (b)

Collis, K. F. A study of concrete and formal operations in school mathematics: A Piagetian viewpoint. Melbourne: Australian Council for Educational Research, 1975.

Collis, K. F. Mathematical thinking in children. In V. P. Varma & P. Williams (Eds.), Piaget, psychology, and education. London: Hodder and Stoughton, 1976.

Collis, K. F. Operational thinking in elementary mathematics. In J. Keats, K. F. Collis, & G. S. Halford (Eds.), Cognitive development. Chichester: Wiley, 1978.

Collis, K. F. School mathematics and stages of development. In R. Modgil & S. Modgil (Eds.), Toward a theory of psychological development. Slough: NFER, 1980. (a)

Collis, K. F. Levels of cognitive functioning and selected curriculum areas. In J. B. Biggs & J. Kirby (Eds.), Cognition, development, and instruction. New York: Academic Press, 1980. (b)

Collis, K. F. The structure of learned outcomes: A refocusing for mathematics learning. In T. P. Carpenter, J. M. Moser, & T. A. Romberg (Eds.), Addition and subtraction: A cognitive perspective. Hillsdale, N.J.: Erlbaum, 1982.

Collis, K. F., & Biggs, J. B. *Classroom examples of cognitive development phenomena: The SOLO taxonomy* (Technical Report prepared for ERDC). Hobart: University of Tasmania, 1979.

Cookson, C., & Moser, J. M. *Coordinated study individual interview procedures* (Working Paper No. 290). Madison: Wisconsin Research and Development Center for Individualized Schooling, 1980.

DeAvila, E., & Havassy, B. *Intelligence of Mexican American children*. Austin, Tex.: Dissemination Center for Bilingual and Bicultural Education, 1974.

DeCorte, E., & Verschaffel, L. Children's solution processes in elementary arithmetic problems: Analysis and improvement. *Journal of Educational Psychology*, 1981, *73*, 765-779.

Flavell, J. H. What is memory development the development of? *Human Development*, 1971, *14*, 272-278.

Fullerton, T. *An approach to number readiness by scalogram analysis*. Unpublished honours thesis, University of Newcastle, 1968.

Ginsburg, H. *Children's arithmetic: The learning process*. New York: Van Nostrand, 1977.

Greeno, J. G. Trends in the theory of knowledge for problem solving. In D. T. Tuma & F. Reif (Eds.), *Problem solving and education: Issues in teaching and research*. Hillsdale, N.J.: Erlbaum, 1980.

Guttman, L. A new approach to factor analysis: The radex. In P. Lagarsfeld (Ed.), *Mathematical thinking in the social sciences*. Glencoe, Ill.: Free Press, 1954.

Hiebert, J. *The effect of cognitive development on first-grade children's abillity to learn linear measurement concepts*

(Technical Report No. 506). Madison: Wisconsin Research and Development Center for Individualized Schooling, 1979.

Inhelder, B., & Piaget, J. The growth of logical thinking from childhood to adolescence. London: Routledge and Kegan Paul, 1958.

Klausmeier, H. J. Origin and overview of IGE. In H. J. Klausmeier, R. Rossmiller, & M. Saily (Eds.), Individually guided elementary education: Concepts and practices. New York: Academic Press, 1977.

Kouba, V. L., & Moser, J. M. Development and validation of curriculum units related to initial sentence writing (Technical Report No. 522). Madison: Wisconsin Research and Development Center for Individualized Schooling, 1979.

Kouba, V. L., & Moser, J. M. Development and validation of curriculum units related to learning of addition and subtraction algorithms (Working Paper No. 287). Madison: Wisconsin Research and Development Center for Individualized Schooling, 1980.

Lovell, K. Intellectual growth and understanding mathematics. Journal for Research in Mathematics Education, 1972, 3, 164-182.

Moser, J. M. Young children's representation of addition and subtraction problems (Technical Paper No. 74). Madison: Wisconsin Research and Development Center for Individualized Schooling, 1979.

Moser, J. M. What is the evidence that children invent problem-solving strategies. In R. Karplus (Eds.), Proceedings of the Fourth International Conference for the Psychology of

Mathematics Education. Berkeley, Calif.: University of California, 1980.

Moser, J. M., & Carpenter, T. P. *Invented processes in solution to arithmetical problems*. Paper presented at the symposium on Children's Arithmetic Beyond Counting. Madison: Wisconsin Center for Education Research, University of Wisconsin, 1982.

Nesher, P., Greeno, J. G., & Riley, M. S. The development of semantic categories for addition and subtraction. *Educational Studies in Mathematics*, 1982, *13*, 373-394.

Pascual-Leone, J. A mathematical model for the transition rule in Piaget's developmental stages. *Acta Psychologica*, 1970, *32*, 301-345.

Pascual-Leone, J. A view of cognition from a formalist's perspective. In K. F. Riegel & J. Meacham (Eds.), *The developing individual in a changing world*. The Hague: Mouton, 1976.

Piaget, J. Forward. In B. Inhelder, H. Sinclair, & M. Bovet (Eds.), *Learning and the development of cognition*. London: Routledge & Kegan Paul, 1974.

Resnick, L. B. *The role of invention in the development of mathematical competence*. Paper presented at the workshop on Children's Mathematical Learning. Learning Research and Development Center, Pittsburgh, Penn., September, 1978.

Romberg, T. A. Developing Mathematical Processes: The elementary mathematics program for individually guided education. In H. J. Klausmeier, R. Rossmiller, & M. Saily (Eds.), *Individually guided elementary education: Concepts and practices*. New York: Academic Press, 1977.

Romberg, T. A. An emerging paradigm for research on addition and subtraction skills. In T. P. Carpenter, J. M. Moser, & T. A. Romberg (Eds.), Addition and subtraction: A cognitive perspective. Hillsdale, N.J.: Erlbaum, 1982.

Romberg, T. A., & Braswell, J. Achievement monitoring via item sampling: A practical data-gathering procedure for formative evaluation. Journal for Research in Mathematics Education, 1973, 4, 262-270.

Romberg, T. A., Carpenter, T. P., & Moser, J. M. Studies in mathematics (Technical Proposal 1978-1979). Madison: Wisconsin Research and Development Center for Individualized Schooling, 1978.

Romberg, T. A., & Collis, K. F. The assessment of children's M-space (Technical Report No. 540). Madison: Wisconsin Research and Development Center for Individualized Schooling, 1980. (a)

Romberg, T. A., & Collis, K. F. The assessment of children's cognitive processing capabilities (Technical Report No. 539). Madison: Wisconsin Research and Development Center for Individualized Schooling, 1980. (b)

Romberg, T. A., Collis, K. F., & Buchanan, A. E. Performance on addition and subtraction problems: Results from individual interviews--Sandy Bay study (Technical Report No. 580). Madison: Wisconsin Research and Development Center for Individualized Schooling, 1981.

Romberg, T. A., Collis, K. F., & Buchanan, A. E. Performance on addition and subtraction problems: Results from achievement *monitoring tests--Sandy Bay study* (Working Paper No. 325). Madison: Wisconsin Center for Education Research, 1982.

Romberg, T. A., Collis, K. F., Buchanan, A. E., & Romberg, M. N. *Classroom learning: Results from observations--Sandy Bay study* (Working Paper No. 326). Madison: Wisconsin Center for Education Research, 1982.

Romberg, T. A., Harvey, J., Moser, J. M., & Montgomery, M. *Developing mathematical processes*. Chicago: Rand McNally, 1974, 1975, 1976.

Romberg, T. A., Small, M., & Carnahan, R. *Research on teaching from a curricular perspective* (Theoretical Paper No. 81). Madison: Wisconsin Research and Development Center for Individualized Schooling, 1979.

Romberg, T. A., Small, M., Carnahan, R., & Cookson, C. *Observer's manual, coordinated study #1, 1978-1980*. Madison: Wisconsin Research and Development Center for Individualized Schooling, 1979.

Thorndike, E. *The psychology of arithmetic*. New York: Macmillan, 1922.

Vergnaud, G. A classification of cognitive tasks and operations of thought involved in addition and subtraction problems. In T. P. Carpenter, J. M. Moser, & T. A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective*. Hillsdale, N.J.: Erlbaum, 1982.

Wohlwill, J. A study of the development of the number concept by scalogram analysis. *Journal of Genetic Psychology*, 1960, *97*, 345-377.

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